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Problem 1 A pair (a, b) of positive integers is good if $\gcd(a, b) = 1$ and for each pair of sets A, B of positive integers such that A, B are, respectively, complete residues system modulo a, b, there are $x \in A, y \in B$ such that $\gcd(x + y, ab) = 1$. For each pair of positive integers a, k, let $f(N)$ the number of $b \leq N$ such b has k distinct prime factors and (a, b) is good. Prove that

$$\liminf_{n\to\infty} f(n)/\frac{n}{(\log n)^k} \geq e^k$$

Problem 2 Let ABC be a triangle and Ω its circumcircle. Let the internal angle bisectors of $\angle BAC, \angle ABC, \angle BCA$ intersect BC, CA, AB on D, E, F, respectively. The perpendicular line to EF through D intersects EF on X and AD intersects EF on Z. The circle internally tangent to Ω and tangent to AB, AC touches Ω on Y. Prove that (XYZ) is tangent to Ω.

Problem 3 positive real C is $n-vengeful$ if it is possible to color the cells of an $n \times n$ chessboard such that:

i) There is an equal number of cells of each color.
ii) In each row or column, at least Cn cells have the same color.

a) Show that $\frac{3}{4}$ is $n-vengeful$ for infinitely many values of n.

b) Show that it does not exist n such that $\frac{4}{5}$ is $n-vengeful$.

Problem 4 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_1 = 1$. For each $n \geq 1$, a_{n+1} is the smallest positive integer, distinct from $a_1, a_2, ..., a_n$, such that $\gcd(a_{n+1}a_n + 1, a_i) = 1$ for each $i = 1, 2, ..., n$. Prove that every positive integer appears in $\{a_n\}_{n=1}^{\infty}$.

Problem 5 Prove that there exists a positive integer $x < 5^{2022}$ such that

$$\{\varphi\sqrt[3]{x}\} < \varphi^{-2022}.$$
