Hong Kong TST for IMO 2022

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by LLL2019

Test 1 Test 1

1. Let a_n be the sequence of rational numbers defined by $a_0 = 2021$, and $a_{n+1} = a_n + \frac{2}{a_n}$, for all $n \geq 0$. Can any $a_n, n \geq 1$, be a square of a rational number?

2. Let a, b, c, d be the roots of the equation $x^4 + x + 1 = 0$. Let $a^5 + 2a + 1, b^5 + 2b + 1, c^5 + 2c + 1, d^5 + 2d + 1$ be roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$. Find the value of $p + 2q + 4r + 8s$.

3. Four non-overlapping families dine at a restaurant together. Each family consists of one father, one mother and three children. There are five tables, which are red, blue, yellow, green and brown respectively. Every table can accomodate four people, and every person must sit at some table. How many ways are there to assign the 20 people to tables so that each child sits at the same table as at least one of his/her parents? (We do not distinguish between the four seats at the same table.)

4. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Diagonals AC and BD meet at E. F and G are points on segments AB and CD respectively. Suppose $AF = 15$, $FB = 13$, $BE = 30$, $ED = 13$, $DG = 7.5$, and $GC = 6.5$. Let P be a point such that $PF \perp AB$ and $PG \perp CD$. Find $\frac{PE}{PO}$.

5. Let $f(n) = \left| \binom{n}{0} + \binom{n}{3} + \cdots + \binom{n}{3[\frac{n}{3}]} - \frac{2^n}{3} \right|$, where $[x]$ is the greatest integer not exceeding x. Find $f(1) + f(2) + \cdots + f(2021)$.

6. There is a set of n 01-sequences of length 200. Every pair of 01-sequences differ at least at 101 positions. (For example, the two 01-sequences of length 6, 111100 and 010001 differ at four positions, 1st, 3rd, 4th and 6th positions, counting from the left.) Is it possible that $n \geq 101$?

Test 2 Test 2

1. Let ABC be a triangle. Let M be the midpoint of BC, and let G be the centroid of $\triangle ABC$. Let D be a point on the segment GM. A straight line passing through D meets the sides AB and AC at E and F respectively (with $E, F \neq A$). Show that

$$[BEGF] + [CFGD] \geq \frac{4}{9}[ABC].$$

When does the equality hold? (Here $[WXYZ]$ is the area of the polygon $WXYZ$, etc.)

2. Let $P(x)$ be a polynomial with integer coefficients. Define a sequence a_n by $a_0 = 0$ and $a_n = P(a_{n-1})$ for all $n \geq 1$. Prove that if there exists a positive integer m for which $a_m = 0$, then $a_1 = 0$ or $a_2 = 0$.

3. Let S be the set of all integers of the form $x^2 + 3xy + 8y^2$. where x and y are integers.
(a) Show that if u and v are in S, then so is uv.
(b) Can an integer of the form $23k + 7$, with k an integer, belong to S?

4. In a chess tournament there are 100 players. On each day of the tournament, each player is designated to be 'white', 'black' or 'idle', and each 'white' player will play a game against every 'black' player. (You may assume that all games fixed for the day can be finished within that day.) At the end of the tournament, it was found that any two players have met exactly once. What is the minimum duration of days that the tournament lasts?

Test 3 Test 3

1. Let n be an integer, and let A be a subset of $\{0, 1, 2, 3, ..., 5^n\}$ consisting of $4n + 2$ numbers. Prove that there exist $a, b, c \in A$ such that $a < b < c$ and $c + 2a > 3b$.

2. Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, ..., d_k)$ such that for every $i = 1, 2, \ldots, k$, the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

3. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

4. Let n and k be two integers with $n > k \geq 1$. There are $2n + 1$ students standing in a circle. Each student S has $2k$ neighbours—namely, the k students closest to S on the right, and the k students closest to S on the left. Suppose that $n + 1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.