Peru IMO TST 2021

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by gnoka, YII.I.

Day1

P1 For any positive integer n, we define $S(n)$ to be the sum of its digits in the decimal representation. Prove that for any positive integer m, there exists a positive integer n such that $S(n) - S(n^2) > m$.

P2 In an acute triangle ABC, its inscribed circle touches the sides AB, BC at the points C_1, A_1 respectively. Let M be the midpoint of the side AC, N be the midpoint of the arc ABC on the circumcircle of triangle ABC, and P be the projection of M on the segment A_1C_1.

Prove that the points P, N and the incenter I of the triangle ABC lie on the same line.

P3 For any positive integer n, we define

$$S_n = \sum_{k=1}^{n} \frac{2^k}{k^2}.$$

Prove that there are no polynomials P, Q with real coefficients such that for any positive integer n, we have $\frac{S_{n+1}}{S_n} = \frac{P(n)}{Q(n)}$.

P4 2020 IMOSL C1

Day2

P1 Find all positive integers m for which there exist three positive integers a, b, c such that $abcm = 1 + a^2 + b^2 + c^2$.

P2 2020 IMOSL C2

P3 Suppose the function $f : [1, +\infty) \to [1, +\infty)$ satisfies the following two conditions:
(i) $f(f(x)) = x^2$ for any $x \geq 1$;
(ii) $f(x) \leq x^2 + 2021x$ for any $x \geq 1$.

1. Prove that $x < f(x) < x^2$ for any $x \geq 1$.

2. Prove that there exists a function f satisfies the above two conditions and the following one:
(iii) There are no real constants c and A, such that $0 < c < 1$, and $\frac{f(x)}{x^2} < c$ for any $x > A$.

Day 3

P1 Suppose positive real numers x, y, z, w satisfy $(x^3+y^3)^4 = z^3+w^3$. Prove that

$$x^4z + y^4w \geq zw.$$

P2 For any positive integers a, b, c, n, we define

$$D_n(a,b,c) = \gcd(a+b+c, a^2+b^2+c^2, a^n+b^n+c^n).$$

1. Prove that if n is a positive integer not divisible by 3, then for any positive integer k, there exist three integers a, b, c such that $\gcd(a,b,c) = 1$, and $D_n(a,b,c) > k$.

2. For any positive integer n divisible by 3, find all values of $D_n(a,b,c)$, where a, b, c are three positive integers such that $\gcd(a,b,c) = 1$.

P3 2020 IMOSL G5
