IMC 2022

www.artofproblemsolving.com/community/c3096331

by Tintarn, Assassino9931

–	Day 1 (August 3)
1	Let $f:[0,1]\to(0,\infty)$ be an integrable function such that $f(x)f(1-x)=1$ for all $x\in[0,1]$. Prove that $\int_0^1 f(x)dx\geq 1$.
2	For a positive integer n determine all $n\times n$ real matrices A which have only real eigenvalues and such that there exists an integer $k\geq n$ with $A+A^k=A^T$.
3	Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3)=3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.
4	Let $n>3$ be an integer. Let Ω be the set of all triples of distinct elements of $\{1,2,\ldots,n\}$. Let m denote the minimal number of colours which suffice to colour Ω so that whenever $1\leq a<b<c<d\leq n$, the triples $\{a,b,c\}$ and $\{b,c,d\}$ have different colours. Prove that $\frac{1}{100}\log\log n\leq m\leq 100\log\log n$.
–	Day 2 (August 4)
5	We colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same colour. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there?
6	Let $p\geq 3$ be a prime number. Prove that there is a permutation $(x_1,\ldots,x_{p-1})$ of $(1,2,\ldots,p-1)$ such that $x_1x_2+x_2x_3+\cdots+x_{p-2}x_{p-1}\equiv 2\pmod p$.
7	Let $A_1,\ldots,A_k$ be $n\times n$ idempotent complex matrices such that $A_iA_j=-A_iA_j$ for all $1\leq i<j\leq k$. Prove that at least one of the matrices has rank not exceeding $\frac{n}{k}$.

8 Let $n, k \geq 3$ be integers, and let S be a circle. Let n blue points and k red points be chosen uniformly and independently at random on the circle S. Denote by F the intersection of the convex hull of the red points and the convex hull of the blue points. Let m be the number of vertices of the convex polygon F (in particular, $m = 0$ when F is empty). Find the expected value of m.