South African Mathematics Olympiad 2022

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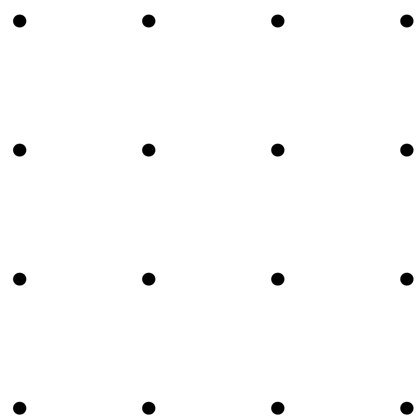
1 Consider 16 points arranged as shown, with horizontal and vertical distances of 1 between consecutive rows and columns. In how many ways can one choose four of these points such that the distance between every two of those four points is strictly greater than 2?



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2 Find all pairs of real numbers x and y which satisfy the following equations:

$$x^2 + y^2 - 48x - 29y + 714 = 0$$
$$2xy - 29x - 48y + 756 = 0$$

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3 Let a, b, and c be nonzero integers. Show that there exists an integer k such that

$$gcd\,(a + kb, c) = gcd\,(a, b, c)$$

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4 Let ABC be a triangle with $AB < AC$. A point P on the circumcircle of ABC (on the same side of BC as A) is chosen in such a way that $BP = CP$. Let BP and the angle bisector of $\angle BAC$ intersect at Q, and let the line through Q and parallel to BC intersect AC at R. Prove that $BR = CR$.

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5 Let $n \geq 3$ be an integer, and consider a set of n points in three-dimensional space such that:

- every two distinct points are connected by a string which is either red, green, blue, or yellow;

- for every three distinct points, if the three strings between them are not all of the same colour, then they are of three different colours;
- not all the strings have the same colour.

Find the maximum possible value of n.

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6 Show that there are infinitely many polynomials P with real coefficients such that if x, y, and z are real numbers such that $x^2 + y^2 + z^2 + 2xyz = 1$, then

$$P\left(x\right)^2 + P\left(y\right)^2 + P\left(z\right)^2 + 2P\left(x\right)P\left(y\right)P\left(z\right) = 1$$
