China Girls Math Olympiad 2022

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–	Day 1
1	Consider all the real sequences $x_0,x_1,\cdots,x_{100}$ satisfying the following two requirements: (1)$x_0=0$; (2)For any integer $i,1\leq i\leq 100$,we have $1\leq x_i-x_{i-1}\leq 2$. Find the greatest positive integer $k\leq 100$,so that for any sequence $x_0,x_1,\cdots,x_{100}$ like this,we have $$x_k+x_{k+1}+\cdots+x_{100}\geq x_0+x_1+\cdots+x_{k-1}.$$
2	Let n be a positive integer. There are $3n$ women's volleyball teams in the tournament, with no more than one match between every two teams (there are no ties in volleyball). We know that there are $3n^2$ games played in this tournament. Proof: There exists a team with at least $\frac{n}{4}$ win and $\frac{n}{4}$ loss
3	In triangle $ABC, AB > AC, I$ is the incenter, AM is the midline. The line crosses I and is perpendicular to BC intersect AM at point L, and the symmetry of I with respect to point A is J Prove that: $\angle ABJ = \angle LBI$.
4	Given a prime number $p\ge 5$. Find the number of distinct remainders modulus p of the product of three consecutive positive integers.
–	Day 2
5	Two points K and L are chosen inside triangle ABC and a point D is chosen on the side AB. Suppose that B, K, L, C are concyclic, $\angle AKD = \angle BCK$ and $\angle ALD = \angle BCL$. Prove that $AK = AL$.
6	Find all integers n satisfying the following property. There exist nonempty finite integer sets A and B such that for any integer m, exactly one of these three statements below is true: (a) There is $a \in A$ such that $m \equiv a \pmod n$, (b) There is $b \in B$ such that $m \equiv b \pmod n$, and (c) There are $a \in A$ and $b \in B$ such that $m \equiv a + b \pmod n$.
7	Let $n\geqslant 3$ be integer. Given convex $n-$polygon $\mathcal{P}$. A $3-$coloring of the vertices of $\mathcal{P}$ is called *nice* such that every interior point of $\mathcal{P}$ is inside or on the bound of a triangle formed by polygon

vertices with pairwise distinct colors. Determine the number of different nice colorings.
(*Two colorings are different as long as they differ at some vertices.*)

8 Let $x_1, x_2, \dots, x_{11}$ be nonnegative reals such that their sum is 1. For $i = 1, 2, \dots, 11$, let

$$y_i = \begin{cases} x_i + x_{i+1}, & i \text{ odd}, \\ x_i + x_{i+1} + x_{i+2}, & i \text{ even}, \end{cases}$$

where $x_{12} = x_1$. And let $F(x_1, x_2, \dots, x_{11}) = y_1 y_2 \dots y_{11}$.
Prove that $x_6 < x_8$ when F achieves its maximum.