Czech And Slovak Mathematical Olympiad, Round III, Category A, 1955

www.artofproblemsolving.com/community/c3103166

by byk7

1 Consider a trapezoid $ABCD, AB \parallel CD, AB > CD$. Let us denote intersections of lines as follows: $E = AC \cap BD, F = AD \cap BC$. Let GH be a line such that $G \in AD, H \in BC, E \in GH, GH \parallel AB$. Moreover, denote K, L midpoints of the bases AB, CD respectively. Show that
(a) the points K, L lie on the line EF,
(b) lines AC, KH and BD, KG are not parallel (denote $M = AC \cap KH, N = BD \cap KG$),
(c) the points F, M, N are collinear.

---

2 Let $\mathsf{S}_1, \mathsf{S}_2$ be concentric spheres with radii a, b respectively, where $a < b$. Denote $ABCDA'B'C'D'$ a square cuboid ($ABCD, A'B'C'D$ are the squares and $AA' \parallel BB' \parallel CC' \parallel DD'$) such that $A, B, C, D \in \mathsf{S}_2$ and the plane $A'B'C'D'$ is tangent to S_1. Finally assume that

$$\frac{AB}{AA'} = \frac{a}{b}.$$

Compute the lengths AB, AA'. How many of such cuboids exist (up to a congruence)?

---

3 In the complex plane consider the unit circle with the origin as its center. Furthermore, consider inscribed regular 17-gon with one of its vertices being $1 + 0i$. How many of its vertices lie in the (open) unit disc centered in $\sqrt{3/2}(1 + i)$?

---

4 Given that a, b, c are distinct real numbers, show that the equation

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = 0$$

has a real root.
