Hong Kong Team Selection Test 2017

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Test 1 August 2016

1 In $\triangle ABC$, let AD be the angle bisector of $\angle BAC$, with D on BC. The perpendicular from B to AD intersects the circumcircle of $\triangle ABD$ at B and E. Prove that E, A and the circumcenter O of $\triangle ABC$ are collinear.

2 In a committee there are n members. Each pair of members are either friends or enemies. Each committee member has exactly three enemies. It is also known that for each committee member, an enemy of his friend is automatically his own enemy. Find all possible value(s) of n

3 Let $f(x)$ be a monic cubic polynomial with $f(0)=-64$, and all roots of $f(x)$ are non-negative real numbers. What is the largest possible value of $f(-1)$? (A polynomial is monic if its leading coefficient is 1.)

4 Consider the sequences with 2016 terms formed by the digits 1, 2, 3, and 4. Find the number of those sequences containing an even number of the digit 1.

5 Find the first digit after the decimal point of the number $\frac{1}{1009}+\frac{1}{1010}+\cdots+\frac{1}{2016}$

6 Given infinite sequences $a_1,a_2,a_3,\cdots$ and $b_1,b_2,b_3,\cdots$ of real numbers satisfying $a_{n+1}+b_{n+1}=\frac{a_n+b_n}{2}$ and $a_{n+1}b_{n+1}=\sqrt{a_nb_n}$ for all $n\geq 1$. Suppose $b_{2016}=1$ and $a_1>0$. Find all possible values of a_1

Test 2 October 2016

1 Given that $\{a_n\}$ is a sequence of integers satisfying the following condition for all positive integral values of n: $a_n+a_{n+1}=2a_{n+2}a_{n+3}+2016$. Find all possible values of a_1 and a_2

2 Let $ABCDEF$ be a convex hexagon such that $\angle ACE=\angle BDF$ and $\angle BCA=\angle EDF$. Let $A_1=AC\cap FB$, $B_1=BD\cap AC$, $C_1=CE\cap BD$, $D_1=DF\cap CE$, $E_1=EA\cap DF$, and $F_1=FB\cap EA$. Suppose B_1,C_1,D_1,F_1 lie on the same circle Γ. The circumcircles of $\triangle BB_1F_1$ and ED_1F_1 meet at F_1 and P. The line F_1P meets Γ again at Q. Prove that B_1D_1 and QC_1 are parrallel. (Here, we use $l_1\cap l_2$ to denote the intersection point of lines l_1 and l_2)

3 Let G be a simple graph with n vertices and m edges. Two vertices are called *neighbours* if there is an edge between them. It turns out the G does not contain any cycles of length from 3 to $2k$ (inclusive), where $k \geq 2$ is a given positive integer.

a) Prove that it is possible to pick a non-empty set S of vertices of G such that every vertex in S has at least $\lceil \frac{m}{n} \rceil$ neighbours that are in S. ($\lceil x \rceil$ denotes the smallest integer larger than or equal to x.)

b) Suppose a set S as described in (a) is chosen. Let H be the graph consisting of the vertices in S and the edges between those vertices only. Let v be a vertex of H. Prove that at least $\left\lceil \left(\frac{m}{n} - 1\right)^k \right\rceil$ vertices of H can be reached by starting at v and travelling across the edges of H for at most k steps. (Note that v itself satisfies this condition, since it can be reached by starting at v and travelling along the edges of H for 0 steps.)

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4 Let n be a positive integer with the following property: $2^n - 1$ divides a number of the form $m^2 + 81$, where m is a positive integer. Find all possible n.

Note There are 2 more tests between test 2 and 3 that are accounted for in the IMO selection process, namely the CHKMO and the APMO.

Test 3 May 1, 2017

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1 a) Do there exist 5 circles in the plane such that each circle passes through exactly 3 centers of other circles?

b) Do there exist 6 circles in the plane such that each circle passes through exactly 3 centers of other circles?

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2 Suppose all of the 200 integers lying in between (and including) 1 and 200 are written on a blackboard. Suppose we choose exactly 100 of these numbers and circle each one of them. By the *score* of such a choice, we mean the square of the difference between the sum of the circled numbers and the sum of the non-circled numbers. What is the average scores over all possible choices for 100 numbers?

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3 At a mathematical competition n students work on 6 problems each one with three possible answers. After the competition, the Jury found that for every two students the number of the problems, for which these students have the same answers, is 0 or 2. Find the maximum possible value of n.

Test 4 May 3, 2017

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1 Decide if there is a permutation $a_1, a_2, \cdots, a_{6666}$ of the numbers $1, 2, \cdots, 6666$ with the property that the sum $k + a_k$ is a perfect square for all $k = 1, 2, \cdots, 6666$

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2 Two circles ω_1 and ω_2, centered at O_1 and O_2, respectively, meet at points A and B. A line through B intersects ω_1 again at C and ω_2 again at D. The tangents to ω_1 and ω_2 at C and D, respectively, meet at E, and the line AE intersects the circle ω through AO_1O_2 at F. Prove that the length of segment EF is equal to the diameter of ω.

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3 Let a sequence of real numbers $a_0, a_1, a_2, \cdots$ satisfies the condition:

$$\sum_{n=0}^{m} a_n \cdot (-1)^n \cdot \binom{m}{n} = 0$$

for all sufficiently large values of m. Show that there exists a polynomial P such that $a_n = P(n)$ for all $n \geq 0$
