National Math Olympiad (3rd Round) 2022

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– Day 1

1 Find all functions $f:\mathbb{R}^+\rightarrow\mathbb{R}^+$ such that for all $x,y,z\in\mathbb{R}^+$

$$f(x+f(y)+f(f(z)))=z+f(y+f(x))$$

2 In the triangle ABC, variable points D,E,F are on the sides[lines] BC,CA,AB respectively so the triangle DFE is similar to the triangle ABC in this order. Circumcircles of BDF and CDE intersect respectively the circumcircle of ABC at P and Q for the second time. Prove that the circumcircle of DPQ passes through a fixed point.

3 We have many three-element subsets of a 1000-element set. We know that the union of every 5 of them has at least 12 elements. Find the most possible value for the number of these subsets.

– Day 2

4 $a_1,a_2,...$ is a sequence of nonzero integer numbers that for all $n\in\mathbb{N}$, if $a_n=2^\alpha k$ such that k is an odd integer and α is a nonnegative integer then: $a_{n+1}=2^\alpha-k$. Prove that if this sequence is periodic, then for all $n\in\mathbb{N}$ we have: $a_{n+2}=a_n$. (The sequence $a_1,a_2,...$ is periodic iff there exists natural number d that for all $n\in\mathbb{N}$ we have: $a_{n+d}=a_n$)

5 Ali has 100 cards with numbers $1,2,\dots,100$. Ali and Amin play a game together. In each step, first Ali chooses a card from the remaining cards and Amin decides to pick that card for himself or throw it away. In the case that he picks the card, he can't pick the next card chosen by Amin, and he has to throw it away. This action repeats until when there is no remaining card for Ali.
Amin wants to pick cards in a way that the sum of the number of his cards is maximized and Ali wants to choose cards in a way that the sum of the number of Amin's cards is minimized. Find the most value of k such that Amin can play in a way that is sure the sum of the number of his cards will be at least equal to k.

6 Prove that among any 9 distinct real numbers, there exist 4 distinct numbers a,b,c,d such that

$$(ac+bd)^2\geq\frac{9}{10}(a^2+b^2)(c^2+d^2)$$

– Geometry

1 Triangle ABC is assumed. The point T is the second intersection of the symmedian of vertex A with the circumcircle of the triangle ABC and the point $D \neq A$ lies on the line AC such that $BA = BD$. The line that at D tangents to the circumcircle of the triangle ADT, intersects the circumcircle of the triangle DCT for the second time at K. Prove that $\angle BKC = 90^\circ$(The symmedian of the vertex A, is the reflection of the median of the vertex A through the angle bisector of this vertex).

2 Constant points B and C lie on the circle ω. The point middle of BC is named M by us. Assume that A is a variable point on the ω and H is the orthocenter of the triangle ABC. From the point H we drop a perpendicular line to MH to intersect the lines AB and AC at X and Y respectively. Prove that with the movement of A on the ω, the orthocenter of the triangle AXY also moves on a circle.

3 The point M is the middle of the side BC of the acute-angled triangle ABC and the points E and F are respectively perpendicular foot of M to the sides AC and AB. The points X and Y lie on the plane such that $\triangle XEC \sim \triangle CEY$ and $\triangle BYF \sim \triangle XBF$(The vertices of triangles with this order are corresponded in the similarities) and the points E and F don't[neither] lie on the line XY. Prove that $XY \perp AM$.

– Algebra

1 We call polynomial $S(x) \in \mathbb{R}[x]$ sadeh whenever it's divisible by x but not divisible by x^2.
For the polynomial $P(x) \in \mathbb{R}[x]$ we know that there exists a sadeh polynomial $Q(x)$ such that $P(Q(x)) - Q(2x)$ is divisible by x^2. Prove that there exists sadeh polynomial $R(x)$ such that $P(R(x)) - R(2x)$ is divisible by x^{1401}.

2 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$:

$$0 \leq y + f(x) - f^{f(y)}(x) \leq 1$$

that here

$$f^n(x) = \underbrace{f(f(\ldots(f}_{n}(x))\ldots)$$

3 Prove that for natural number n it's possible to find complex numbers $\omega_1, \omega_2, \cdots, \omega_n$ on the unit circle that

$$\left|\sum_{j=1}^{n} \omega_j\right| = \left|\sum_{j=1}^{n} \omega_j^2\right| = n - 1$$

iff $n = 2$ occurs.

– Number Theory

1 Assume natural number $n \geq 2$. Amin and Ali take turns playing the following game:
In each step, the player whose turn has come chooses index i from the set $\{0, 1, \cdots, n\}$, such that none of the two players had chosen this index in the previous turns; also this player in this turn chooses nonzero rational number a_i too. Ali performs the first turn. The game ends when all the indices $i \in \{0, 1, \cdots, n\}$ were chosen. In the end, from the chosen numbers the following polynomial is built:

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

Ali's goal is that the preceding polynomial has a rational root and Amin's goal is that to prevent this matter.
Find all $n \geq 2$ such that Ali can play in a way to be sure independent of how Amin plays achieves his goal.

2 For two rational numbers r, s we say:

$$r \mid s$$

whenever there exists $k \in \mathbb{Z}$ such that:

$$s = kr$$

$(a_n)_{n\in\mathbb{N}}$ is an increasing sequence of pairwise coprime natural numbers and $(b_n)_{n\in\mathbb{N}}$ is a sequence of distinct natural numbers. Assume that for all $n \in \mathbb{N}$ we have:

$$\sum_{i=1}^{n} \frac{1}{a_i} \mid \sum_{i=1}^{n} \frac{1}{b_i}$$

Prove that **for all** $n \in \mathbb{N}$ we have: $a_n = b_n$.

3 We call natural number m **ziba**, iff every natural number n with the condition $1 \leq n \leq m$ can be shown as sum of [some of] positive and distinct divisors of m. Prove that infinitely ziba numbers in the form of $(k \in \mathbb{N}) k^2 + k + 2022$ exist.

– Combinatorics

1 For each natural number k find the least number n such that in every tournament with n vertices, there exists a vertex with in-degree and out-degree at least k.
(Tournament is directed complete graph.)

2 $m \times n$ grid is tiled by mosaics 2×2 and 1×3 (horizontal and vertical). Prove that the number of ways to choose a 1×2 rectangle (horizontal and vertical) such that one of its cells is tiled by 2×2 mosaic and the other cell is tiled by 1×3 mosaic [horizontal and vertical] is an even number.

3 We have $n \geq 3$ points on the plane such that no three are collinear. Prove that it's possible to name them $P_1, P_2, \ldots, P_n$ such that for all $1 < i < n$, the angle $\angle P_{i-1}P_iP_{i+1}$ is acute.
