Lexington Math Tournament

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by parmenides51

–	Team Round
1	Derek and Jacob have a cake in the shape a rectangle with dimensions 14 inches by 9 inches. They make a deal to split it: Derek takes home the portion of the cake that is less than one inch from the border, while Jacob takes home the remainder of the cake. Let $D : J$ be the ratio of the amount of cake Derek took to the amount of cake Jacob took, where D and J are relatively prime positive integers. Find $D + J$.
2	Five people are standing in a straight line, and the distance between any two people is a unique positive integer number of units. Find the least possible distance between the leftmost and rightmost people in the line in units.
3	Let the four real solutions to the equation $x^2 + \frac{144}{x^2} = 25$ be r_1, r_2, r_3, and r_4. Find $\|r_1\| + \|r_2\| + \|r_3\| + \|r_4\|$.
4	Jeff has a deck of 12 cards: 4 Ls, 4 Ms, and 4 Ts. Armaan randomly draws three cards without replacement. The probability that he takes 3 Ls can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
5	Find the sum $$\sum_{n=1}^{2020} \gcd(n^3 - 2n^2 + 2021, n^2 - 3n + 3).$$
6	For all y, define cubic $f_y(x)$ such that $f_y(0) = y$, $f_y(1) = y + 12$, $f_y(2) = 3y^2$, $f_y(3) = 2y + 4$. For all y, $f_y(4)$ can be expressed in the form $ay^2 + by + c$ where a, b, c are integers. Find $a + b + c$.
7	Kevin has a square piece of paper with creases drawn to split the paper in half in both directions, and then each of the four small formed squares diagonal creases drawn, as shown below. `https://cdn.artofproblemsolving.com/attachments/2/2/70d6c54e86856af3a977265a8054fd9b0444b` `png` Find the sum of the corresponding numerical values of figures below that Kevin can create by folding the above piece of paper along the creases. (The figures are to scale.) Kevin cannot cut the paper or rip it in any way. `https://cdn.artofproblemsolving.com/attachments/a/c/e0e62a743c00d35b9e6e2f702106016b9e787` `png`

8 The 53-digit number

$$37,984,318,966,591,152,105,649,545,470,741,788,308,402,068,827,142,719$$

can be expressed as n^21 where n is a positive integer. Find n.

9 Let $r_1, r_2, ..., r_{2021}$ be the not necessarily real and not necessarily distinct roots of $x^{2022}+2021x = 2022$. Let $S_i = r_i^{2021} + 2022r_i$ for all $1 \le i \le 2021$. Find $\left|\sum_{i=1}^{2021} S_i\right| = |S_1 + S_2 + ... + S_{2021}|$.

10 In a country with 5 distinct cities, there may or may not be a road between each pair of cities. It's possible to get from any city to any other city through a series of roads, but there is no set of three cities $\{A, B, C\}$ such that there are roads between A and B, B and C, and C and A. How many road systems between the five cities are possible?

– Accuracy Round

1 Kevin colors a ninja star on a piece of graph paper where each small square has area 1 square inch. Find the area of the region colored, in square inches.
`https://cdn.artofproblemsolving.com/attachments/3/3/86f0ae7465e99d3e4bd3a816201383b98dc42`
`png`

2 Let $a♠b = \frac{a^2-b^2}{2b-2a}$. Given that $3♠x = -10$, compute x.

3 Find the difference between the greatest and least values of $lcm(a, b, c)$, where a, b, and c are distinct positive integers between 1 and 10, inclusive.

4 Kevin runs uphill at a speed that is 4 meters per second slower than his speed when he runs downhill. Kevin takes a total of 80 seconds to run up and down a hill on one path. Given that the path is 300 meters long (he travels 600 meters total), find how long Kevin takes to run up the hill in seconds.

5 A bag contains 5 identical blue marbles and 5 identical green marbles. In how many ways can 5 marbles from the bag be arranged in a row if each blue marble must be adjacent to at least 1 green marble?

6 Jacob likes to watchMickeyMouse Clubhouse! One day, he decides to create his own Mickey-Mouse head shown below, with two circles ω_1 and ω_2 and a circle ω, and centers O_1, O_2, and O, respectively. Let ω_1 and ω meet at points P_1 and Q_1, and let ω_2 and ω meet at points P_2 and Q_2. Point P_1 is closer to O_2 than Q_1, and point P_2 is closer to O_1 than Q_2. Given that P_1 and P_2 lie on O_1O_2 such that $O_1P_1 = P_1P_2 = P_2O_2 = 2$, and $Q_1O_1 \parallel Q_2O_2$, the area of ω can be written as $n\pi$. Find n.

https://cdn.artofproblemsolving.com/attachments/6/d/d98a05ee2218e80fd84d299d47201669736d9
png

7 A teacher wishes to separate her 12 students into groups. Yesterday, the teacher put the students into 4 groups of 3. Today, the teacher decides to put the students into 4 groups of 3 again. However, she doesn't want any pair of students to be in the same group on both days. Find how many ways she could formthe groups today.

8 A ray originating at point P intersects a circle with center O at points A and B, with $PB > PA$. Segment $\overline{OP}$ intersects the circle at point C. Given that $PA = 31$, $PC = 17$, and $\angle PBO = 60^o$, find the radius of the circle.

9 A rook is randomly placed on an otherwise empty 8×8 chessboard. Owen makes moves with the rook by randomly choosing 1 of the 14 possible moves. Find the expected value of the number of moves it takes Owen to move the rook to the top left square. Note that a rook can move any number of squares either in the horizontal or vertical direction each move.

10 In a room, there are 100 light switches, labeled with the positive integers $1, 2, ..., 100$. They're all initially turned off. On the i th day for $1 \le i \le 100$, Bob flips every light switch with label number k divisible by i a total of $\frac{k}{i}$ times. Find the sum of the labels of the light switches that are turned on at the end of the 100th day.

Tie Let L be the number of times the letter L appeared on the Speed Round, M be the number of times the letter M appeared on the Speed Round, and T be the number of times the letter T appeared on the Speed Round. Find the value of LMT.

– Guts Round

– Round 1

p1. A box contains 1 ball labelledW, 1 ball labelled E, 1 ball labelled L, 1 ball labelled C, 1 ball labelled O, 8 balls labelled M, and 1 last ball labelled E. One ball is randomly drawn from the box. The probability that the ball is labelled E is $\frac{1}{a}$. Find a.

p2. Let
$$G + E + N = 7$$
$$G + E + O = 15$$
$$N + T = 22.$$
Find the value of $T + O$.

p3. The area of $\triangle LMT$ is 22. Given that $MT = 4$ and that there is a right angle at M, find the length of LM.

Round 2

p4. Kevin chooses a positive 2-digit integer, then adds 6 times its unit digit and subtracts 3 times its tens digit from itself. Find the greatest common factor of all possible resulting numbers.

p5. Find the maximum possible number of times circle D can intersect pentagon $GRASS'$ over all possible choices of points G, R, A, S, and S'.

p6. Find the sum of the digits of the integer solution to $(\log_2 x) \cdot (\log_4 \sqrt{x}) = 36$.

Round 3

p7. Given that x and y are positive real numbers such that $x^2 + y = 20$, the maximum possible value of $x + y$ can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

p8. In $\triangle DRK$, $DR = 13$, $DK = 14$, and $RK = 15$. Let E be the point such that $ED = ER = EK$. Find the value of $\lfloor DE + RE + KE \rfloor$.

p9. Subaru the frog lives on lily pad 1. There is a line of lily pads, numbered 2, 3, 4, 5, 6, and 7. Every minute, Subaru jumps from his current lily pad to a lily pad whose number is either 1 or 2 greater, chosen at random from valid possibilities. There are alligators on lily pads 2 and 5. If Subaru lands on an alligator, he dies and time rewinds back to when he was on lily pad number 1. Find how many times Subaru is expected to die before he reaches pad 7.

Round 4

p10. Find the sum of the following series:

$$\sum_{i=1}^{\infty} = \frac{\sum_{j=1}^{i} j}{2^i} = \frac{1}{2^1} + \frac{1+2}{2^2} + \frac{1+2+3}{2^3} + \frac{1+2+3+4}{2^4} + ...$$

p11. Let $\phi(x)$ be the number of positive integers less than or equal to x that are relatively prime to x. Find the sum of all x such that $\phi(\phi(x)) = x - 3$. Note that 1 is relatively prime to every positive integer.

p12. On a piece of paper, Kevin draws a circle. Then, he draws two perpendicular lines. Finally, he draws two perpendicular rays originating from the same point (an L shape). What is the maximum number of sections into which the lines and rays can split the circle?

Round 5

p13. In quadrilateral $ABCD$, $\angle A = 90^o$, $\angle C = 60^o$, $\angle ABD = 25^o$, and $\angle BDC = 5^o$. Given that $AB = 4\sqrt{3}$, the area of quadrilateral $ABCD$ can be written as $a\sqrt{b}$. Find $10a + b$.

p14. The value of

$$\sum_{n=2}^{6}\left(\frac{n^4+1}{n^4-1}\right) - 2\sum_{n=2}^{6}\left(\frac{n^3-n^2+n}{n^4-1}\right)$$

can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $100m + n$.

p15. Positive real numbers x and y satisfy the following 2 equations.

$$x^{1+x^{1+x^{1+\dots}}} = 8$$

$$\sqrt[24]{y + \sqrt[24]{y + \sqrt[24]{y + \dots}}} = x$$

Find the value of $\lfloor y \rfloor$.

PS. You should use hide for answers. Rounds 6-9 have been posted here (https://artofproblemsolving.com/community/c3h3167130p28823260). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 6

p16. Given that x and y are positive real numbers such that $x^3 + y = 20$, the maximum possible value of $x + y$ can be written as $\frac{a\sqrt{b}}{c}$ +d where a, b, c, and d are positive integers such that $gcd(a, c) = 1$ and b is square-free. Find $a + b + c + d$.

p17. In $\triangle\ DRK$, $DR = 13$, $DK = 14$, and $RK = 15$. Let E be the intersection of the altitudes of $\triangle\ DRK$. Find the value of $\lfloor DE + RE + KE \rfloor$.

p18. Subaru the frog lives on lily pad 1. There is a line of lily pads, numbered 2, 3, 4, 5, 6, and 7. Every minute, Subaru jumps from his current lily pad to a lily pad whose number is either 1 or

2 greater, chosen at random from valid possibilities. There are alligators on lily pads 2 and 5. If Subaru lands on an alligator, he dies and time rewinds back to when he was on lily pad number 1. Find the expected number of jumps it takes Subaru to reach pad 7.

Round 7

This set has problems whose answers depend on one another.

p19. Let B be the answer to Problem 20 and let C be the answer to Problem 21. Given that

$$f(x) = x^3 - Bx - C = (x-r)(x-s)(x-t)$$

where r, s, and t are complex numbers, find the value of $r^2 + s^2 + t^2$.

p20. Let A be the answer to Problem 19 and let C be the answer to Problem 21. Circles ω_1 and ω_2 meet at points X and Y . Let point $P \ne Y$ be the point on ω_1 such that PY is tangent to ω_2, and let point $Q \ne Y$ be the point on ω_2 such that QY is tangent to ω_1. Given that $PX = A$ and $QX = C$, find XY .

p21. Let A be the answer to Problem 19 and let B be the answer to Problem 20. Given that the positive difference between the number of positive integer factors of A^B and the number of positive integer factors of B^A is D, and given that the answer to this problem is an odd prime, find $\frac{D}{B} - 40$.

Round 8

p22. Let $v_p(n)$ for a prime p and positive integer n output the greatest nonnegative integer x such that p^x divides n. Find

$$\sum_{i=1}^{50} \sum_{p=1}^{i} \binom{v_p(i)+1}{2},$$

where the inner summation only sums over primes p between 1 and i .

p23. Let a, b, and c be positive real solutions to the following equations.

$$\frac{2b^2+2c^2-a^2}{4} = 25$$

$$\frac{2c^2+2a^2-b^2}{4} = 49$$

$$\frac{2a^2+2b^2-c^2}{4} = 64$$

The area of a triangle with side lengths a, b, and c can be written as $\frac{x\sqrt{y}}{z}$ where x and z are relatively prime positive integers and y is square-free. Find $x+y+z$.

p24. Alan, Jiji, Ina, Ryan, and Gavin want to meet up. However, none of them know when to go, so they each pick a random 1 hour period from 5 AM to 11 AM to meet up at Alan's house. Find the probability that there exists a time when all of them are at the house at one time.

Round 9

p25. Let n be the number of registered participantsin this LMT. Estimate the number of digits of $\left[\binom{n}{2}\right]$ in base 10. If your answer is A and the correct answer is C, then your score will be

$$\left\lfloor \max\left(0, 20-\left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

p26. Let γ be theminimum value of x^x over all real numbers x. Estimate $\lfloor 10000\gamma \rfloor$. If your answer is A and the correct answer is C, then your score will be

$$\left\lfloor \max\left(0, 20-\left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

p27. Let

$$E = \log_{13} 1 + log_{13}2 + log_{13}3 + ... + log_{13}513513.$$

Estimate $\lfloor E \rfloor$. If your answer is A and the correct answer is C, your score will be

$$\left\lfloor \max\left(0, 20-\left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

PS. You should use hide for answers. Rounds 1-5 have been posted here (`https://artofproblemsolving.com/community/c3h3167127p28823220`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Speed

– posted for the image link