

Middle European Mathematical Olympiad 2022

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– Individual Competition

- 1 Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + f(x + y)) = x + f(f(x) + y)$$

holds for all real numbers x and y .

- 2 Let n be a positive integer. Anna and Beatrice play a game with a deck of n cards labelled with the numbers $1, 2, \dots, n$. Initially, the deck is shuffled. The players take turns, starting with Anna. At each turn, if k denotes the number written on the topmost card, then the player first looks at all the cards and then rearranges the k topmost cards. If, after rearranging, the topmost card shows the number k again, then the player has lost and the game ends. Otherwise, the turn of the other player begins. Determine, depending on the initial shuffle, if either player has a winning strategy, and if so, who does.
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- 3 Let $ABCD$ be a parallelogram with $\angle DAB < 90^\circ$. Let E be the point on the line BC such that $AE = AB$ and let F be the point on the line CD such that $AF = AD$. The circumcircle of the triangle CEF intersects the line AE again in P and the line AF again in Q . Let X be the reflection of P over the line DE and Y the reflection of Q over the line BF . Prove that A, X, Y lie on the same line.
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- 4 Initially, two distinct positive integers a and b are written on a blackboard. At each step, Andrea picks two distinct numbers x and y on the blackboard and writes the number $\gcd(x, y) + \text{lcm}(x, y)$ on the blackboard as well. Let n be a positive integer. Prove that, regardless of the values of a and b , Andrea can perform a finite number of steps such that a multiple of n appears on the blackboard.
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– Team Competition

- 1 Given a pair (a_0, b_0) of real numbers, we define two sequences a_0, a_1, a_2, \dots and b_0, b_1, b_2, \dots of real numbers by $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n b_n$ for all $n = 0, 1, 2, \dots$. Find all pairs (a_0, b_0) of real numbers such that $a_{2022} = a_0$ and $b_{2022} = b_0$.
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- 2 Let k be a positive integer and a_1, a_2, \dots, a_k be nonnegative real numbers. Initially, there is a sequence of $n \geq k$ zeros written on a blackboard. At each step, Nicole chooses k consecutive numbers written on the blackboard and increases the first number by a_1 , the second one by a_2 , and so on, until she increases the k -th one by a_k . After a positive number of steps, Nicole

managed to make all the numbers on the blackboard equal. Prove that all the nonzero numbers among a_1, a_2, \dots, a_k are equal.

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- 3** Let n be a positive integer. There are n purple and n white cows queuing in a line in some order. Tim wishes to sort the cows by colour, such that all purple cows are at the front of the line. At each step, he is only allowed to swap two adjacent groups of equally many consecutive cows. What is the minimal number of steps Tim needs to be able to fulfill his wish, regardless of the initial alignment of the cows?
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- 4** Let n be a positive integer. We are given a $2n \times 2n$ table. Each cell is coloured with one of $2n^2$ colours such that each colour is used exactly twice. Jana stands in one of the cells. There is a chocolate bar lying in one of the other cells. Jana wishes to reach the cell with the chocolate bar. At each step, she can only move in one of the following two ways. Either she walks to an adjacent cell or she teleports to the other cell with the same colour as her current cell. (Jana can move to an adjacent cell of the same colour by either walking or teleporting.) Determine whether Jana can fulfill her wish, regardless of the initial configuration, if she has to alternate between the two ways of moving and has to start with a teleportation.
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- 5** Let Ω be the circumcircle of a triangle ABC with $\angle CAB = 90^\circ$. The medians through B and C meet Ω again at D and E , respectively. The tangent to Ω at D intersects the line AC at X and the tangent to Ω at E intersects the line AB at Y . Prove that the line XY is tangent to Ω .
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- 6** Let $ABCD$ be a convex quadrilateral such that $AC = BD$ and the sides AB and CD are not parallel. Let P be the intersection point of the diagonals AC and BD . Points E and F lie, respectively, on segments BP and AP such that $PC = PE$ and $PD = PF$. Prove that the circumcircle of the triangle determined by the lines AB, CD, EF is tangent to the circumcircle of the triangle ABP .
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- 7** Determine all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that f is increasing (not necessarily strictly) and the numbers $f(n) + n + 1$ and $f(f(n)) - f(n)$ are both perfect squares for every positive integer n .
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- 8** We call a positive integer *cheesy* if we can obtain the average of the digits in its decimal representation by putting a decimal separator after the leftmost digit. Prove that there are only finitely many *cheesy* numbers.
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