Czech-Polish-Slovak Junior Match 2022

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by parmenides51

– Individual

1 Let $n \geq 3$. Suppose $a_1, a_2, ..., a_n$ are n distinct in pairs real numbers.
In terms of n, find the smallest possible number of different assumed values by the following n numbers:

$$a_1 + a_2, a_2 + a_3, ..., a_{n-1} + a_n, a_n + a_1$$

2 Solve the following system of equations in integer numbers:

$$\begin{cases} x^2 = yz + 1 \\ y^2 = zx + 1 \\ z^2 = xy + 1 \end{cases}$$

3 Given is a convex pentagon $ABCDE$ in which $\angle A = 60^o$, $\angle B = 100^o$, $\angle C = 140^o$.
Show that this pentagon can be placed in a circle with a radius of $\frac{2}{3}AD$.

4 Let a and b be positive integers with the property that $\frac{a}{b} > \sqrt{2}$. Prove that

$$\frac{a}{b} - \frac{1}{2ab} > \sqrt{2}$$

5 An integer $n \geq 1$ is *good* if the following property is satisfied:
If a positive integer is divisible by each of the nine numbers $n + 1, n + 2, ..., n + 9$, this is also divisible by $n + 10$.
How many good integers are $n \geq 1$?

– Team

1 Determine the largest possible value of the expression $ab + bc + 2ac$ for non-negative real numbers a, b, c whose sum is 1.

2 The number 2022 is written on the board. In each step, we replace one of the 2 digits with the number 2022.
For example

$$2022 \Rightarrow 2020222 \Rightarrow 2020220222 \Rightarrow ...$$

After how many steps can a number divisible by 22 be written on the board? Specify all options.

3 The points D, E, F lie respectively on the sides BC, CA, AB of the triangle ABC such that $FB = BD$, $DC = CE$, and the lines EF and BC are parallel. Tangent to the circumscribed circle of triangle DEF at point F intersects line AD at point P. Perpendicular bisector of segment EF intersects the segment AC at Q. Show that the lines PQ and BC are parallel.

4 Find all triples (a, b, c) of integers that satisfy the equations

$a + b = c$ and $a^2 + b^3 = c^2$

5 Given a regular nonagon $A_1A_2A_3A_4A_5A_6A_7A_8A_9$ with side length 1. Diagonals A_3A_7 and A_4A_8 intersect at point P. Find the length of segment PA_1.

6 Find all integers $n \geq 4$ with the following property:
Each field of the $n \times n$ table can be painted white or black in such a way that each square of this table had the same color as exactly the two adjacent squares. (Squares are adjacent if they have exactly one side in common.)
How many different colorings of the 6×6 table fields meet the above conditions?