

**Czech-Polish-Slovak Match 2022**

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– Day 1

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**1** Let  $k \leq 2022$  be a positive integer. Alice and Bob play a game on a  $2022 \times 2022$  board. Initially, all cells are white. Alice starts and the players alternate. In her turn, Alice can either color one white cell in red or pass her turn. In his turn, Bob can either color a  $k \times k$  square of white cells in blue or pass his turn. Once both players pass, the game ends and the person who colored more cells wins (a draw can occur). For each  $1 \leq k \leq 2022$ , determine which player (if any) has a winning strategy.

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**2** Find all functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $f(f(x) + \frac{y+1}{f(y)}) = \frac{1}{f(y)} + x + 1$  for all  $x, y > 0$ .  
*Proposed by Dominik Burek, Poland*

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**3** Circles  $\Omega_1$  and  $\Omega_2$  with different radii intersect at two points, denote one of them by  $P$ . A variable line  $l$  passing through  $P$  intersects the arc of  $\Omega_1$  which is outside of  $\Omega_2$  at  $X_1$ , and the arc of  $\Omega_2$  which is outside of  $\Omega_1$  at  $X_2$ . Let  $R$  be the point on segment  $X_1X_2$  such that  $X_1P = RX_2$ . The tangent to  $\Omega_1$  through  $X_1$  meets the tangent to  $\Omega_2$  through  $X_2$  at  $T$ . Prove that line  $RT$  is tangent to a fixed circle, independent of the choice of  $l$ .

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– Day 2

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**4** Find all positive integers  $n$ , such that  $\sigma(n) = \tau(n) \lceil \sqrt{n} \rceil$ .

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**5** Let  $ABC$  be a triangle with  $AB < AC$  and circumcenter  $O$ . The angle bisector of  $\angle BAC$  meets the side  $BC$  at  $D$ . The line through  $D$  perpendicular to  $BC$  meets the segment  $AO$  at  $X$ . Furthermore, let  $Y$  be the midpoint of segment  $AD$ . Prove that points  $B, C, X, Y$  are concyclic.

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**6** Consider 26 letters  $A, \dots, Z$ . A string is a finite sequence consisting of those letters. We say that a string  $s$  is nice if it contains each of the 26 letters at least once, and each permutation of letters  $A, \dots, Z$  occurs in  $s$  as a subsequences the same number of times. Prove that:  
(a) There exists a nice string.  
(b) Any nice string contains at least 2022 letters.

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