Middle European Mathematical Olympiad 2016

www.artofproblemsolving.com/community/c313036

by gavrilos, danepale

– Individual Competition

1 Let $n \geq 2$ be an integer, and let $x_1, x_2, \ldots, x_n$ be reals for which:

(a) $x_j > -1$ for $j = 1, 2, \ldots, n$ and

(b) $x_1 + x_2 + \ldots + x_n = n$.

Prove that

$$\sum_{j=1}^{n} \frac{1}{1+x_j} \geq \sum_{j=1}^{n} \frac{x_j}{1+x_j^2}$$

and determine when does the equality occur.

2 There are $n \geq 3$ positive integers written on a board. A *move* consists of choosing three numbers a, b, c written from the board such that there exists a non-degenerate non-equilateral triangle with sides a, b, c and replacing those numbers with $a+b-c, b+c-a$ and $c+a-b$.

Prove that a sequence of moves cannot be infinite.

3 Let ABC be an acute triangle such that $\angle BAC > 45^\circ$ with circumcenter O. A point P is chosen inside triangle ABC such that A, P, O, B are concyclic and the line BP is perpendicular to the line CP. A point Q lies on the segment BP such that the line AQ is parallel to the line PO.

Prove that $\angle QCB = \angle PCO$.

4 Find all $f : \mathbb{N} \to \mathbb{N}$ such that $f(a) + f(b)$ divides $2(a + b - 1)$ for all $a, b \in \mathbb{N}$.

Remark: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of the positive integers.

– Team Competition

1 Find all triples (a, b, c) of real numbers such that

$$a^2 + ab + c = 0,$$

$$b^2 + bc + a = 0,$$

$$c^2 + ca + b = 0.$$

2 Let $\mathbb{R}$ denote the set of the reals. Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = xf(f(y-x)) + xf(2x) + f(x^2)$$

for all real x, y.

---

3 A 8×8 board is given, with sides directed north-south and east-west.
It is divided into 1×1 cells in the usual manner. In each cell, there is most one *house*. A house occupies only one cell.

A house is *in the shade* if there is a house in each of the cells in the south, east and west sides of its cell. In particular, no house placed on the south, east or west side of the board is in the shade.

Find the maximal number of houses that can be placed on the board such that no house is in the shade.

---

4 An exam was taken by some students. Each problem was worth 1 point for the correct answer, and 0 points for an incorrect one.

For each question, at least one student answered it correctly. Also, there are two students with different scores on the exam.

Prove that there exists a question for which the following holds:
The average score of the students who answered the question correctly is greater than the average score of the students who didn't.

---

5 Let ABC be an acute triangle for which $AB \neq AC$, and let O be its circumcenter. Line AO meets the circumcircle of ABC again in D, and the line BC in E. The circumcircle of CDE meets the line CA again in P. The lines PE and AB intersect in Q. Line passing through O parallel to the line PE intersects the A-altitude of ABC in F.

Prove that $FP = FQ$.

---

6 Let ABC be a triangle for which $AB \neq AC$. Points K, L, M are the midpoints of the sides BC, CA, AB.
The incircle of ABC with center I is tangent to BC in D. A line passing through the midpoint of ID perpendicular to IK meets the line LM in P.

Prove that $\angle PIA = 90^\circ$.

---

7 A positive integer n is *Mozart* if the decimal representation of the sequence $1, 2, \ldots, n$ contains each digit an even number of times.

Prove that:
1. All Mozart numbers are even.
2. There are infinitely many Mozart numbers.

---

8 For a positive integer n, the equation $a^2 + b^2 + c^2 + n = abc$ is given in the positive integers.

Prove that:
1. There does not exist a solution (a, b, c) for $n = 2017$.
2. For $n = 2016$, a is divisible by 3 for all solutions (a, b, c).
3. There are infinitely many solutions (a, b, c) for $n = 2016$.