

**Tuymaada Olympiad 2022**

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– Juniors

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– Day 1

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**1** Arnim and Brentano have a little vase with 1500 candies on the table and a huge sack with spare candies under the table. They play a game taking turns, Arnim begins. At each move a player can either eat 7 candies or take 6 candies from under the table and add them to the vase. A player cannot go under the table in two consecutive moves. A player is declared the winner if he leaves the vase empty. In any other case, if a player cannot make a move in his turn, the game is declared a tie. Is there a winning strategy for one of the players?

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**2** Given are integers  $a, b, c$  and an odd prime  $p$ . Prove that  $p$  divides  $x^2 + y^2 + ax + by + c$  for some integers  $x$  and  $y$ .  
(A. Golovanov)

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**3** Bisectors of a right triangle  $\triangle ABC$  with right angle  $B$  meet at point  $I$ . The perpendicular to  $IC$  drawn from  $B$  meets the line  $IA$  at  $D$ ; the perpendicular to  $IA$  drawn from  $B$  meets the line  $IC$  at  $E$ . Prove that the circumcenter of the triangle  $\triangle IDE$  lies on the line  $AC$ .  
(A. Kuznetsov)

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**4** Several *good* points, several *bad* points and several segments are drawn in the plane. Each segment connects a *good* point and a *bad* one; at most 100 segments begin at each point. We have paint of 200 colors. One half of each segment is painted with one of these colors, and the other half with another one. Is it always possible to do it so that every two segments with common end are painted with four different colors?  
(M. Qi, X. Zhang)

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– Day 2

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**5** Each row of a  $24 \times 8$  table contains some permutation of the numbers  $1, 2, \dots, 8$ . In each column the numbers are multiplied. What is the minimum possible sum of all the products?  
(C. Wu)

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**6** The city of Neverreturn has  $N$  bus stops numbered  $1, 2, \dots, N$ . Each bus route is one-way and has only two stops, the beginning and the end. The route network is such that departing from any stop one cannot return to it using city buses. When the mayor notices a route going from a stop with a greater number to a stop with a lesser number, he orders to exchange the number

plates of its beginning and its end. Can the plate changing go on infinitely?  
(K. Ivanov)

- 7  $M$  is the midpoint of the side  $AB$  in an equilateral triangle  $\triangle ABC$ . The point  $D$  on the side  $BC$  is such that  $BD : DC = 3 : 1$ . On the line passing through  $C$  and parallel to  $MD$  there is a point  $T$  inside the triangle  $\triangle ABC$  such that  $\angle CTA = 150$ . Find the  $\angle MTD$ .  
(K. Ivanov)

- 8 Eight poles stand along the road. A sparrow starts at the first pole and once in a minute flies to a neighboring pole. Let  $a(n)$  be the number of ways to reach the last pole in  $2n + 1$  flights (we assume  $a(m) = 0$  for  $m < 3$ ). Prove that for all  $n \geq 4$

$$a(n) - 7a(n-1) + 15a(n-2) - 10a(n-3) + a(n-4) = 0.$$

(T. Amdeberhan, F. Petrov)

– Seniors

– Day 1

- 1 Some of 100 towns of a kingdom are connected by roads. It is known that for each two towns  $A$  and  $B$  connected by a road there is a town  $C$  which is not connected by a road with at least one of the towns  $A$  and  $B$ . Determine the maximum possible number of roads in the kingdom.

- 2 Two circles  $w_1$  and  $w_2$  of different radii touch externally at  $L$ . A line touches  $w_1$  at  $A$  and  $w_2$  at  $B$  (the points  $A$  and  $B$  are different from  $L$ ). A point  $X$  is chosen in the plane.  $Y$  and  $Z$  are the second points of intersection of the lines  $XA$  and  $XB$  with  $w_1$  and  $w_2$  respectively. Prove that all  $X$  such that  $AB \parallel YZ$  belong to one circle.

- 3 Is there a colouring of all positive integers in three colours so that for each positive integer the numbers of its divisors of any two colours differ at most by 2?

- 4 For every positive  $a_1, a_2, \dots, a_6$ , prove the inequality

$$\sqrt[4]{\frac{a_1}{a_2 + a_3 + a_4}} + \sqrt[4]{\frac{a_2}{a_3 + a_4 + a_5}} + \dots + \sqrt[4]{\frac{a_6}{a_1 + a_2 + a_3}} \geq 2$$

– Day 2

- 5 Prove that a quadratic trinomial  $x^2 + ax + b$  ( $a, b \in R$ ) cannot attain at ten consecutive integral points values equal to powers of 2 with non-negative integral exponent.  
(F. Petrov)

- 6** Kostya marked the points  $A(0, 1), B(1, 0), C(0, 0)$  in the coordinate plane. On the legs of the triangle  $ABC$  he marked the points with coordinates  $(\frac{1}{2}, 0), (\frac{1}{3}, 0), \dots, (\frac{1}{n+1}, 0)$  and  $(0, \frac{1}{2}), (0, \frac{1}{3}), \dots, (0, \frac{1}{n+1})$ . Then Kostya joined each pair of marked points with a segment. Sasha drew a  $1 \times n$  rectangle and joined with a segment each pair of integer points on its border. As a result both the triangle and the rectangle are divided into polygons by the segments drawn. Who has the greater number of polygons:  
Sasha or Kostya?  
(*M. Alekseyev*)
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- 7** A  $1 \times 5n$  rectangle is partitioned into tiles, each of the tile being either a separate  $1 \times 1$  square or a broken domino consisting of two such squares separated by four squares (not belonging to the domino). Prove that the number of such partitions is a perfect fifth power.  
(*K. Kokhas*)
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- 8** In an acute triangle  $\triangle ABC$  the points  $C_m, A_m, B_m$  are the midpoints of  $AB, BC, CA$  respectively. Inside the triangle  $\triangle ABC$  a point  $P$  is chosen so that  $\angle PCB = \angle B_m BC$  and  $\angle PAB = \angle ABB_m$ . A line passing through  $P$  and perpendicular to  $AC$  meets the median  $BB_m$  at  $E$ . Prove that  $E$  lies on the circumcircle of the triangle  $\triangle A_m B_m C_m$ .  
(*K. Ivanov*)
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