South East Mathematical Olympiad 2022

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–	Grade 10
–	Day 1
1	The positive sequence $\{a_n\}$ satisfies:$a_1 = 1+\sqrt{2}$ and $(a_n - a_{n-1})(a_n + a_{n-1} - 2\sqrt{n}) = 2(n \geq 2)$. (1)Find the general formula of $\{a_n\}$; (2)Find the set of all the positive integers n so that $\lfloor a_n \rfloor = 2022$.
2	In acute triangle ABC AB¿AC. H is the orthocenter. M is midpoint of BC and AD is the symmedian line. Prove that if $\angle ADH = \angle MAH$, EF bisects segment AD. `https://s2.loli.net/2022/08/02/t9xzTV8IEv1qQRm.jpg`
3	If x_i is an integer greater than 1, let $f(x_i)$ be the greatest prime factor of $x_i, x_{i+1} = x_i - f(x_i)$ ($i \geq 0$ and i is an integer). (1) Prove that for any integer x_0 greater than 1, there exists a natural number$k(x_0)$, such that $x_{k(x_0)+1} = 0$ Grade 10: (2) Let $V_{(x_0)}$ be the number of different numbers in $f(x_0), f(x_1), \cdots, f(x_{k(x_0)})$. Find the largest number in $V(2), V(3), \cdots, V(781)$ and give reasons. Note: Bai Lu Zhou Academy was founded in 1241 and has a history of 781 years. Grade 11: (2) Let $V_{(x_0)}$ be the number of different numbers in $f(x_0), f(x_1), \cdots, f(x_{k(x_0)})$. Find the largest number in $V(2), V(3), \cdots, V(2022)$ and give reasons.
4	Given $m, n \geq 2$.Paint each cell of a $m \times n$ board S red or blue so that:for any two red cells in a row,one of the two columns they belong to is all red,and the other column has at least one blue cell in it.Find the number of ways to paint S like this.
–	Day 2
5	Positive sequences $\{a_n\}, \{b_n\}$ satisfy:$a_1 = b_1 = 1, b_n = a_n b_{n-1} - \frac{1}{4}(n \geq 2)$. Find the minimum value of $4\sqrt{b_1 b_2 \cdots b_m} + \sum_{k=1}^{m} \frac{1}{a_1 a_2 \cdots a_k}$,where m is a given positive integer.
6	H is the orthocenter of $\triangle ABC$,the circle with center H passes through A,and it intersects with AC, AB at two other points D, E.The orthocenter of $\triangle ADE$ is H',line AH' intersects with DE at point F.Point P is inside the quadrilateral $BCDE$,so that $\triangle PDE \sim \triangle PBC$.Let point K be the intersection of line HH' and line PF.Prove that A, H, P, K lie on one circle. `https://i.ibb.co/mcyhxRM/graph.jpg`

7 Let a, b be positive integers.Prove that there are no positive integers on the interval $\left[\frac{b^2}{a^2+ab}, \frac{b^2}{a^2+ab-1}\right)$.

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8 Tao plays the following game:given a constant $v > 1$;for any positive integer m,the time between the m^{th} round and the $(m+1)^{th}$ round of the game is 2^{-m} seconds;Tao chooses a circular safe area whose radius is 2^{-m+1} (with the border,and the choosing time won't be calculated) on the plane in the m^{th} round;the chosen circular safe area in each round will keep its center fixed,and its radius will decrease at the speed v in the rest of the time(if the radius decreases to 0,erase the circular safe area);if it's possible to choose a circular safe area inside the union of the rest safe areas sometime before the 100^{th} round(including the 100^{th} round),then Tao wins the game.If Tao has a winning strategy,find the minimum value of $\left\lfloor \frac{1}{v-1} \right\rfloor$.

– Grade 11

– Day 1

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1 Let x_1, x_2, x_3 be three positive real roots of the equation $x^3 + ax^2 + bx + c = 0$ $(a, b, c \in R)$ and $x_1 + x_2 + x_3 \leq 1$. Prove that

$$a^3(1 + a + b) - 9c(3 + 3a + a^2) \leq 0$$

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2 same as Grade 10 P2

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3 There are n people in line, counting $1, 2, \cdots, n$ from left to right, those who count odd numbers quit the line, the remaining people press 1,2 from right to left, and count off again, those who count odd numbers quit the line, and then the remaining people count off again from left to right$\cdots$ Keep doing that until only one person is in the line. $f(n)$ is the number of the last person left at the first count. Find the expression for $f(n)$ and find the value of $f(2022)$

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4 Similar to Grade 10 P3

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5 Let a, b, c, d be non-negative integers. (1) If $a^2 + b^2 - cd^2 = 2022$, find the minimum of $a + b + c + d$; (1) If $a^2 - b^2 + cd^2 = 2022$, find the minimum of $a + b + c + d$.

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6 Let O be the circumcenter of $\triangle ABC$. A circle with center P pass through A and O and $OP//BC$. D is a point such that $\angle DBA = \angle DCA = \angle BAC$. Prove that: Circle (P), circle (BCD) and the circle with diameter (AD) share a common point.
`https://services.artofproblemsolving.com/download.php?id=YXR0YWNobWVudHMvMS9jLzlmZjdlN2Ex`
`=\&rn=c291dGhlYXN0UDYuanBn`

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7 Prove that for any positive real number λ,there are n positive numbers $a_1, a_2, \cdots, a_n (n \geq 2)$,so that $a_1 < a_2 < \cdots < a_n < 2^n \lambda$ and for any $k = 1, 2, \cdots, n$ we have

$$\gcd(a_1, a_k) + \gcd(a_2, a_k) + \cdots + \gcd(a_n, a_k) \equiv 0 \pmod{a_k}$$

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8 same as Grade 10 P8
