China Second Round 2022

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1 In a convex quadrilateral $ABCD$, $\angle ABC = \angle ADC = 90°$. A point P is chosen from the diagonal BD such that $\angle APB = 2\angle CPD$, points X, Y is chosen from the segment AP such that $\angle AXB = 2\angle ADB$, $\angle AYD = 2\angle ABD$. Prove that: $BD = 2XY$.

2 Integer n has k different prime factors. Prove that $\sigma(n) \mid (2n - k)!$

3 Let $a_1, a_2, \cdots, a_{100}$ be non-negative integers such that (1) There are positive integers $k \le 100$ such that $a_1 \le a_2 \le \cdots \le a_k$
and $a_i = 0 \ (i > k)$;

$(2) \ a_1 + a_2 + a_3 + \cdots + a_{100} = 100$;

$(3) \ a_1 + 2a_2 + 3a_3 + \cdots + 100a_{100} = 2022$.

Find the minimum of $a_1 + 2^2 a_2 + 3^2 a_3 + \cdots + 100^2 a_{100}$.

4 Find the smallest positive integer k with the following property: if each cell of a 100×100 grid is dyed with one color and the number of cells of each color is not more than 104, then there is a $k \times 1$ or $1 \times k$ rectangle that contains cells of at least three different colors.