VIII - Rioplatense Mathematical Olympiad, Level 3 1999

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–	Day 1
1	Let ABC be a scalene acute triangle whose orthocenter is H. M is the midpoint of segment BC. N is the point where the segment AM intersects the circle determined by B, C, and H. Show that lines HN and AM are perpendicular.
2	Let $p_1, p_2, ..., p_k$ be k different primes. We consider all positive integers that use only these primes (not necessarily all) in their prime factorization, and arrange those numbers in increasing order, forming an infinite sequence: $a_1 < a_2 < ... < a_n < ...$ Prove that, for every number c, there exists n such that $a_{n+1} - a_n > c$.
3	Two players A and B play the following game: A chooses a point, with integer coordinates, on the plane and colors it green, then B chooses 10 points of integer coordinates, not yet colored, and colors them yellow. The game always continues with the same rules; A and B choose one and ten uncolored points and color them green and yellow, respectively. a. The objective of A is to achieve 111^2 green points that are the intersections of 111 horizontal lines and 111 vertical lines (parallel to the coordinate axes). B's goal is to stop him. Determine which of the two players has a strategy that ensures you achieve your goal. b. The objective of A is to achieve 4 green points that are the vertices of a square with sides parallel to the coordinate axes. B's goal is to stop him. Determine which of the two players has a strategy that will ensure that they achieve their goal.
–	Day 2
4	Prove the following inequality: $$\frac{1}{\sqrt[3]{1^2}+\sqrt[3]{1\cdot 2}+\sqrt[3]{2^2}}+\frac{1}{\sqrt[3]{3^2}+\sqrt[3]{3\cdot 4}+\sqrt[3]{4^2}}+...+\frac{1}{\sqrt[3]{999^2}+\sqrt[3]{999\cdot 1000}+\sqrt[3]{1000^2}}>\frac{9}{2}$$ (The member on the left has 500 fractions.)
5	The quadrilateral $ABCD$ is inscribed in a circle of radius 1, so that AB is a diameter of the circumference and $CD = 1$. A variable point X moves along the semicircle determined by AB that does not contain C or D. Determine the position of X for which the sum of the distances from X to lines BC, CD and DA is maximum.
6	At a big New Year's Eve party, each guest receives two hats: one red and one blue. At the beginning of the party, all the guests put on the red hat. Several times throughout the evening, the

announcer announces the name of one of the guests and, at that moment, the named and each of his friends change the hat they are wearing for the other color. Show that the announcer can make it so that all the guests are wearing the blue hat when the party is over.

Note: All guests remain at the party from start to finish.