VII - Rioplatense Mathematical Olympiad, Level 3 1998

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– Day 1

1 Consider an arc AB of a circle C and a point P variable in that arc AB. Let D be the midpoint of the arc AP that doeas not contain B and let E be the midpoint of the arc BP that does not contain A. Let C_1 be the circle with center D passing through A and C_2 be the circle with center E passing through B. Prove that the line that contains the intersection points of C_1 and C_2 passes through a fixed point.

2 Given an integer $n > 2$, consider all sequences $x_1, x_2, ..., x_n$ of nonnegative real numbers such that

$$x_1 + 2x_2 + ... + nx_n = 1.$$

Find the maximum value and the minimum value of $x_1^2 + x_2^2 + ... + x_n^2$ and determine all the sequences $x_1, x_2, ..., x_n$ for which these values are obtained.

3 Let X be a finite set of positive integers.
Prove that for every subset A of X, there is a subset B of X, with the following property:
For each element e of X, e divides an odd number of elements of B, if and only if e is an element of A.

– Day 2

4 Let M be a subset of $\{1, 2, ..., 1998\}$ with 1000 elements. Prove that it is always possible to find two elements a and b in M, not necessarily distinct, such that $a + b$ is a power of 2.

5 We say that M is the midpoint of the open polygonal XYZ, formed by the segments XY, YZ, if M belongs to the polygonal and divides its length by half. Let ABC be a acute triangle with orthocenter H. Let $A_1, B_1, C_1, A_2, B_2, C_2$ be the midpoints of the open polygonal $CAB, ABC, BCA, BHC, CHA,$ respectively. Show that the lines A_1A_2, B_1B_2 and C_1C_2 are concurrent.

6 Let k be a fixed positive integer. For each $n = 1, 2, ...,$ we will call *configuration* of order n any set of kn points of the plane, which does not contain 3 collinear, colored with k given colors, so that there are n points of each color. Determine all positive integers n with the following property: in each configuration of order n, it is possible to select three points of each color, such that the k triangles with vertices of the same color that are determined are disjoint in pairs.