IV - Rioplatense Mathematical Olympiad, Level 3 1995

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– Day 1

1 Let n and p be two integers with p positive prime, such that $pn + 1$ is a perfect square. Show that $n + 1$ is the sum of p perfect squares, not necessarily distinct.

2 In a circle of center O and radius r, a triangle ABC of orthocenter H is inscribed. It is considered a triangle $A'B'C'$ whose sides have by length the measurements of the segments AB, CH and $2r$. Determine the triangle ABC so that the area of the triangle $A'B'C'$ is maximum.

3 Given a regular tetrahedron with edge a, its edges are divided into n equal segments, thus obtaining $n + 1$ points: 2 at the ends and $n - 1$ inside. The following set of planes is considered:
• those that contain the faces of the tetrahedron, and • each of the planes parallel to a face of the tetrahedron and containing at least one of the points determined above.
Now all those points P that belong (simultaneously) to four planes of that set are considered.
Determine the smallest positive natural n so that among those points P the eight vertices of a square-based rectangular parallelepiped can be chosen.

– Day 2

4 Given the natural numbers a and b, with $1 \le a < b$, prove that there exist natural numbers $n_1 < n_2 < ... < n_k$, with $k \le a$ such that

$$\frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k}$$

5 Consider $2n$ points in the plane. Two players A and B alternately choose a point on each move.
After $2n$ moves, there are no points left to choose from and the game ends.
Add up all the distances between the points chosen by A and add up all the distances between the points chosen by B. The one with the highest sum wins.
If A starts the game, describe the winner's strategy.

Clarification: Consider that all the partial sums of distances between points give different numbers.

6 A convex polygon with $2n$ sides is called *rhombic* if its sides are equal and all pairs of opposite sides are parallel.
A rhombic polygon can be partitioned into rhombic quadrilaterals.
For what value of$n$, a $2n$-sided rhombic polygon splits into 666 rhombic quadrilaterals?