V - Rioplatense Mathematical Olympiad, Level 3 1996

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by parmenides51

– Day 1

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1 Given a family C of circles of the same radius R, which completely covers the plane (that is, every point in the plane belongs to at least one circle of the family), prove that there exist two circles of the family such that the distance between their centers is less than or equal to $R\sqrt{3}$.

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2 A *magic square* is a table
https://cdn.artofproblemsolving.com/attachments/7/9/3b1e2b2f5d2d4c486f57c4ad68b66f7d7e56c
png
in which all the natural numbers from 1 to 16 appear and such that: • all rows have the same sum s. • all columns have the same sum s. • both diagonals have the same sum s .
It is known that $a_{22} = 1$ and $a_{24} = 2$. Calculate a_{44}.

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3 The real numbers x, y, z, distinct in pairs satisfy

$$\begin{cases} x^2 = 2 + y \\ y^2 = 2 + z \\ z^2 = 2 + x. \end{cases}$$

Find the possible values of $x^2 + y^2 + z^2$.

– Day 2

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4 Let S be the circle of center O and radius R, and let A, A' be two diametrically opposite points in S. Let P be the midpoint of OA' and ℓ a line passing through P, different from AA' and from the perpendicular on AA'. Let B and C be the intersection points of ℓ with S and let M be the midpoint of BC.
a) Let H be the foot of the altitude from A in the triangle ABC. Let D be the intersection point of the line $A'M$ with AH. Determine the locus of point D while ℓ varies .
b) Line AM intersects OD at I. Prove that $2OI = ID$ and determine the locus of point I while ℓ varies .

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5 There is a board with n rows and 4 columns, and white, yellow and light blue chips.
Player A places four tokens on the first row of the board and covers them so Player B doesn't know them.
How should player B do to fill the minimum number of rows with chips that will ensure that in any of the rows he will have at least three hits?

Clarification: A hit by player B occurs when he places a token of the same color and in the same column as A.

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6 Find all integers k for which, there is a function $f : N \to Z$ that satisfies:
(i) $f(1995) = 1996$
(ii) $f(xy) = f(x) + f(y) + kf(m_{xy})$ for all natural numbers x, y,wherem_{xy} denotes the greatest common divisor of the numbers x, y.

Clarification: $N = \{1, 2, 3, ...\}$ and $Z = \{... - 2, -1, 0, 1, 2, ...\}$.
