Germany Team Selection Test 2016

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– VAIMO 1

1 The two circles Γ_1 and Γ_2 with the midpoints O_1 resp. O_2 intersect in the two distinct points A and B. A line through A meets Γ_1 in $C \neq A$ and Γ_2 in $D \neq A$. The lines CO_1 and DO_2 intersect in X.
Prove that the four points O_1, O_2, B and X are concyclic.

2 The positive integers $a_1, a_2, \dots, a_n$ are aligned clockwise in a circular line with $n \geq 5$. Let $a_0 = a_n$ and $a_{n+1} = a_1$. For each $i \in \{1, 2, \dots, n\}$ the quotient

$$q_i = \frac{a_{i-1} + a_{i+1}}{a_i}$$

is an integer. Prove

$$2n \leq q_1 + q_2 + \dots + q_n < 3n.$$

3 In the beginning there are 100 integers in a row on the blackboard. Kain and Abel then play the following game: A *move* consists in Kain choosing a chain of consecutive numbers; the length of the chain can be any of the numbers $1, 2, \dots, 100$ and in particular it is allowed that Kain only chooses a single number. After Kain has chosen his chain of numbers, Abel has to decide whether he wants to add 1 to each of the chosen numbers or instead subtract 1 from of the numbers. After that the next move begins, and so on.

If there are at least 98 numbers on the blackboard that are divisible by 4 after a move, then Kain has won.

Prove that Kain can force a win in a finite number of moves.

– VAIMO 2

1 Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

2 Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

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3 Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.
