Saint Petersburg Mathematical Olympiad 2022

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–	Grade 11
2	Given is a triangle ABC with $\angle BAC = 45$; AD, BE, CF are altitudes and $EF \cap BC = X$. If $AX \parallel DE$, find the angles of the triangle.
3	Ivan and Kolya play a game, Ivan starts. Initially, the polynomial $x-1$ is written of the blackboard. On one move, the player deletes the current polynomial $f(x)$ and replaces it with $ax^{n+1}-f(-x)-2$, where $\deg(f)=n$ and a is a real root of f. The player who writes a polynomial which does not have real roots loses. Can Ivan beat Kolya?
4	We will say that a set of real numbers $A=(a_1,...,a_{17})$ is stronger than the set of real numbers $B=(b_1,...,b_{17})$, and write $A>B$ if among all inequalities $a_i>b_j$ the number of true inequalities is at least 3 times greater than the number of false. Prove that there is no chain of sets $A_1, A_2, ..., A_N$ such that $A_1>A_2>\cdots A_N>A_1$. Remark: For 11.4, the constant 3 is changed to 2 and $N=3$ and 17 is changed to m and n in the definition (the number of elements don't have to be equal).
5	Altitudes AA_1, BB_1, CC_1 of acute triangle ABC intersect at point H. On the tangent drawn from point C to the circle (AB_1C_1), the perpendicular HQ is drawn (the point Q lies inside the triangle ABC). Prove that the circle passing through the point B_1 and touching the line AB at point A is also tangent to line A_1Q.
6	Find all pairs of nonzero rational numbers x, y, such that every positive rational number can be written as $\frac{\{rx\}}{\{ry\}}$ for some positive rational r.
7	Given is a set of $2n$ cards numbered $1, 2, \cdots, n$, each number appears twice. The cards are put on a table with the face down. A set of cards is called good if no card appears twice. Baron Munchausen claims that he can specify 80 sets of n cards, of which at least one is sure to be good. What is the maximal n for which the Baron's words could be true?
–	Grade 10
2	Similar to 11.4
3	Given is a triangle ABC with altitude AH, diameter of the circumcircle AD and incenter I. Prove that $\angle BIH = \angle DIC$.

4 There are two piles of stones: 1703 stones in one pile and 2022 in the other. Sasha and Olya play the game, making moves in turn, Sasha starts. Let before the player's move the heaps contain a and b stones, with $a \geq b$. Then, on his own move, the player is allowed take from the pile with a stones any number of stones from 1 to b. A player loses if he can't make a move. Who wins?

Remark: For 10.4, the initial numbers are $(444, 999)$

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5 Let n be a positive integer and let $a_1, a_2, \cdots a_k$ be all numbers less than n and coprime to n in increasing order. Find the set of values the function $f(n) = gcd(a_1^3 - 1, a_2^3 - 1, \cdots, a_k^3 - 1)$.

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7 Given is a graph G of $n+1$ vertices, which is constructed as follows: initially there is only one vertex v, and one a move we can add a vertex and connect it to exactly one among the previous vertices. The vertices have non-negative real weights such that v has weight 0 and each other vertex has a weight not exceeding the avarage weight of its neighbors, increased by 1. Prove that no weight can exceed n^2.

– Grade 9

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1 The positive integers a and b are such that $a+k$ is divisible by $b+k$ for all positive integers numbers $k < b$. Prove that $a-k$ is divisible by $b-k$ for all positive integers $k < b$.

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2 12 schoolchildren are engaged in a circle of patriotic songs, each of them knows a few songs (maybe none). We will say that a group of schoolchildren can sing a song if at least one member of the group knows it. Supervisor the circle noticed that any group of 10 circle members can sing exactly 20 songs, and any group of 8 circle members - exactly 16 songs. Prove that the group of all 12 circle members can sing exactly 24 songs.

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3 Given is a trapezoid $ABCD$, $AD \parallel BC$. The angle bisectors of the two pairs of opposite angles meet at X, Y. Prove that $AXYD$ and $BXYC$ are cyclic.

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4 We will say that a point of the plane (u, v) lies between the parabolas $y = f(x)$ and $y = g(x)$ if $f(u) \leq v \leq g(u)$. Find the smallest real p for which the following statement is true: for any segment, the ends and the midpoint of which
lie between the parabolas $y = x^2$ and $y = x^2 + 1$, then they lie entirely between the parabolas $y = x^2$ and $y = x^2 + p$.

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5 Similar to 10.4

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7 Given are n distinct natural numbers. For any two of them, the one is obtained from the other by permuting its digits (zero cannot be put in the first place). Find the largest n such that it is possible all these numbers to be divisible by the smallest of them?
