Federal Competition For Advanced Students, Part 2 2022

www.artofproblemsolving.com/community/c3165304

by parmenides51

–	Day 1
1	Find all functions $f : Z_{>0} \to Z_{>0}$ with $a - f(b) \| af(a) - bf(b)$ for all $a, b \in Z_{>0}$. *(Theresia Eisenkoelbl)*
2	Let ABC be an acute-angled, non-isosceles triangle with orthocenter H, M midpoint of side AB and w bisector of angle $\angle ACB$. Let S be the point of intersection of the perpendicular bisector of side AB with w and F the foot of the perpendicular from H on w. Prove that the segments MS and MF are equal. *(Karl Czakler)*
3	Lisa writes a positive whole number in the decimal system on the blackboard and now makes in each turn the following: The last digit is deleted from the number on the board and then the remaining shorter number (or 0 if the number was one digit) becomes four times the number deleted number added. The number on the board is now replaced by the result of this calculation. Lisa repeats this until she gets a number for the first time was on the board. (a) Show that the sequence of moves always ends. (b) If Lisa begins with the number $53^{2022} - 1$, what is the last number on the board? Example: If Lisa starts with the number 2022, she gets $202 + 4 \cdot 2 = 210$ in the first move and overall the result $$2022 \to 210 \to 21 \to 6 \to 24 \to 18 \to 33 \to 15 \to 21$$. Since Lisa gets 21 for the second time, the turn order ends. *(Stephan Pfannerer)*
–	Day 2
4	Decide whether for every polynomial P of degree at least 1, there exist infinitely many primes that divide $P(n)$ for at least one positive integer n. *(Walther Janous)*
5	Let ABC be an isosceles triangle with base AB. We choose a point P inside the triangle on altitude through C. The circle with diameter CP intersects the straight line through B and P

again at the point D_P and the Straight through A and C one more time at point E_P. Prove that there is a point F such that for any choice of P the points D_P, E_P and F lie on a straight line.

(Walther Janous)

---

6 (a) Prove that a square with sides 1000 divided into 31 squares tiles, at least one of which has a side length less than 1.

(b) Show that a corresponding decomposition into 30 squares is also possible.

(Walther Janous)
