National Science Olympiad 2022

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– Day 1

1 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$ we have

$$f(f(f(x)) + f(y)) = f(y) - f(x)$$

2 Let $P(x)$ be a polynomial with integer coefficient such that $P(1) = 10$ and $P(-1) = 22$.
(a) Give an example of $P(x)$ such that $P(x) = 0$ has an integer root.
(b) Suppose that $P(0) = 4$, prove that $P(x) = 0$ does not have an integer root.

3 Let $ABCD$ be a rectangle. Points E and F are on diagonal AC such that F lies between A and E; and E lies between C and F. The circumcircle of triangle BEF intersects AB and BC at G and H respectively, and the circumcircle of triangle DEF intersects AD and CD at I and J respectively. Prove that the lines GJ, IH and AC concur at a point.

4 Given a regular 26-gon. Prove that for any 9 vertices of that regular 26-gon, then there exists three vertices that forms an isosceles triangle.

– Day 2

5 Let $N \geq 2$ be a positive integer. Given a sequence of natural numbers $a_1, a_2, a_3, \ldots, a_{N+1}$ such that for every integer $1 \leq i \leq j \leq N+1$,

$$a_i a_{i+1} a_{i+2} \ldots a_j \not\equiv 1 \mod N$$

Prove that there exist a positive integer $k \leq N+1$ such that $\gcd(a_k, N) \neq 1$

6 In a triangle ABC, D and E lies on AB and AC such that DE is parallel to BC. There exists point P in the interior of $BDEC$ such that

$$\angle BPD = \angle CPE = 90^\circ$$

Prove that the line AP passes through the circumcenter of triangles EPD and BPC.

7 Let A be the sequence of zeroes and ones (binary sequence). The sequence can be modified by the following operation: we may pick a block or a contiguous subsequence where there are an unequal number of zeroes and ones, and then flip their order within the block (so block $a_1, a_2, \ldots, a_r$ becomes $a_r, a_{r-1}, \ldots, a_1$).

As an example, let A be the sequence $1, 1, 0, 0, 1$. We can pick block $1, 0, 0$ and flip it, so the sequence $1, \boxed{1, 0, 0}, 1$ becomes $1, \boxed{0, 0, 1}, 1$. However, we cannot pick block $1, 1, 0, 0$ and flip their order since they contain the same number of 1s and 0s.

Two sequences A and B are called *related* if A can be transformed into B using a finite number the operation mentioned above.

Determine the largest natural number n for which there exists n different sequences $A_1, A_2, \ldots, A_n$ where each sequence consists of 2022 digits, and for every index $i \neq j$, the sequence A_i is not related to A_j.

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8 Determine the smallest positive real K such that the inequality

$$K + \frac{a+b+c}{3} \geq (K+1)\sqrt{\frac{a^2+b^2+c^2}{3}}$$

holds for any real numbers $0 \leq a, b, c \leq 1$.

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