www.artofproblemsolving.com/community/c3168305
by parmenides51

– Team Round

– **p1.** If $x + z = v$, $w + z = 2v$, $z - w = 2y$, and $y \neq 0$, compute the value of
$$\left(x + y + \frac{x}{y}\right)^{101}.$$

p2. Every minute, a snail picks one cardinal direction (either north, south, east, or west) with equal probability and moves one inch in that direction. What is the probability that after four minutes the snail is more than three inches away from where it started?

p3. What is the probability that a point chosen randomly from the interior of a cube is closer to the cube's center than it is to any of the cube's eight vertices?

p4. Let $ABCD$ be a rectangle where $AB = 4$ and $BC = 3$. Inscribe circles within triangles ABC and ACD. What is the distance between the centers of these two circles?

p5. C is a circle centered at the origin that is tangent to the line $x - y\sqrt{3} = 4$. Find the radius of C.

p6. I have a fair 100-sided die that has the numbers 1 through 100 on its sides. What is the probability that if I roll this die three times that the number on the first roll will be greater than or equal to the sum of the two numbers on the second and third rolls?

p7. List all solutions (x, y, z) of the following system of equations with x, y, and z positive real numbers:
$$x^2 + y^2 = 16$$
$$x^2 + z^2 = 4 + xz$$
$$y^2 + z^2 = 4 + yz\sqrt{3}$$

p8. $A_1A_2A_3A_4A_5A_6A_7$ is a regular heptagon (7 sided-figure) centered at the origin where $A_1 = (\sqrt[91]{6}, 0)$. $B_1B_2B_3...B_{13}$ is a regular triskaidecagon (13 sided-figure) centered at the origin where $B_1 = (0, \sqrt[91]{41})$. Compute the product of all lengths A_iB_j , where i ranges between 1 and 7, inclusive, and j ranges between 1 and 13, inclusive.

p9. How many three-digit integers are there such that one digit of the integer is exactly two times a digit of the integer that is in a different place than the first? (For example, 100, 122, and 124 should be included in the count, but 42 and 130 should not.)

p10. Let α and β be the solutions of the quadratic equation

$$x^2 - 1154x + 1 = 0.$$

Find $\sqrt[4]{\alpha} + \sqrt[4]{\beta}$.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Tiebreaker Round

– **p1.** Let $p_b(m)$ be the sum of digits of m when m is written in base b. (So, for example, $p_2(5) = 2$). Let $f(0) = 2007^{2007}$, and for $n \geq 0$ let $f(n+1) = p_7(f(n))$. What is $f(10^{10000})$?

p2. Compute:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4n}{n^4 - 8n^2 + 4}.$$

p3. $ABCDEFGH$ is an octagon whose eight interior angles all have the same measure. The lengths of the eight sides of this octagon are, in some order,

$$2, 2\sqrt{2}, 4, 4\sqrt{2}, 6, 7, 7, \ \ and \ \ 8.$$

Find the area of $ABCDEFGH$.

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– Individual Round

– **p1.** There are 32 balls in a box: 6 are blue, 8 are red, 4 are yellow, and 14 are brown. If I pull out three balls at once, what is the probability that none of them are brown?

p2. Circles A and B are concentric, and the area of circle A is exactly 20% of the area of circle B. The circumference of circle B is 10. A square is inscribed in circle A. What is the area of that square?

p3. If $x^2 + y^2 = 1$ and $x, y \in R$, let q be the largest possible value of $x + y$ and p be the smallest possible value of $x + y$. Compute pq.

p4. Yizheng and Jennifer are playing a game of ping-pong. Ping-pong is played in a series of consecutive matches, where the winner of a match is given one point. In the scoring system that Yizheng and Jennifer use, if one person reaches 11 points before the other person can reach 10 points, then the person who reached 11 points wins. If instead the score ends up being tied 10-to-10, then the game will continue indefinitely until one person's score is two more than the other person's score, at which point the person with the higher score wins. The probability that Jennifer wins any one match is 70% and the score is currently at 9-to-9. What is the probability that Yizheng wins the game?

p5. The squares on an 8×8 chessboard are numbered left-to-right and then from top-to-bottom (so that the top-left square is $\#1$, the top-right square is $\#8$, and the bottom-right square is $\#64$). 1 grain of wheat is placed on square $\#1$, 2 grains on square $\#2$, 4 grains on square $\#3$, and so on, doubling each time until every square of the chessboard has some number of grains of wheat on it. What fraction of the grains of wheat on the chessboard are on the rightmost column?

p6. Let f be any function that has the following property: For all real numbers x other than 0 and 1,

$$f\left(1 - \frac{1}{x}\right) + 2f\left(\frac{1}{1-x}\right) + 3f(x) = x^2.$$

Compute $f(2)$.

p7. Find all solutions of:

$$(x^2 + 7x + 6)^2 + 7(x^2 + 7x + 6) + 6 = x.$$

p8. Let $\triangle\ ABC$ be a triangle where $AB = 25$ and $AC = 29$. C_1 is a circle that has AB as a diameter and C_2 is a circle that has BC as a diameter. D is a point on C_1 so that $BD = 15$ and $CD = 21$. C_1 and C_2 clearly intersect at B; let E be the other point where C_1 and C_2 intersect. Find all possible values of ED.

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– Devil Round

– **p1.** If
$$\begin{cases} a^2+b^2+c^2=1000 \\ (a+b+c)^2=100 \\ ab+bc=10 \end{cases}$$
what is ac?

p2. If a and b are real numbers such that $a \neq 0$ and the numbers 1, $a+b$, and a are, in some order, the numbers 0, $\frac{b}{a}$, and b, what is $b-a$?

p3. Of the first 120 natural numbers, how many are divisible by at least one of 3, 4, 5, 12, 15, 20, and 60?

p4. For positive real numbers a, let p_a and q_a be the maximum and minimum values, respectively, of $\log_a(x)$ for $a \le x \le 2a$. If $p_a - q_a = \frac12$, what is a?

p5. Let ABC be an acute triangle and let a, b, and c be the sides opposite the vertices A, B, and C, respectively. If $a = 2b\sin A$, what is the measure of angle B?

p6. How many ordered triples (x, y, z) of positive integers satisfy the equation
$$x^3 + 2y^3 + 4z^3 = 9?$$

p7. Joe has invented a robot that travels along the sides of a regular octagon. The robot starts at a vertex of the octagon and every minute chooses one of two directions (clockwise or counterclockwise) with equal probability and moves to the next vertex in that direction. What is the probability that after 8 minutes the robot is directly opposite the vertex it started from?

p8. Find the nonnegative integer n such that when
$$\left(x^2 - \frac{1}{x}\right)^n$$
is completely expanded the constant coefficient is 15.

p9. For each positive integer k, let
$$f_k(x) = \frac{kx+9}{x+3}.$$
Compute
$$f_1 \circ f_2 \circ ... \circ f_{13}(2).$$

p10. Exactly one of the following five integers cannot be written in the form $x^2+y^2+5z^2$, where x, y, and z are integers. Which one is it?

$$2003, 2004, 2005, 2006, 2007$$

p11. Suppose that two circles C_1 and C_2 intersect at two distinct points M and N. Suppose that P is a point on the line MN that is outside of both C_1 and C_2. Let A and B be the two distinct points on C_1 such that AP and BP are each tangent to C_1 and B is inside C_2. Similarly, let D and E be the two distinct points on C_2 such that DP and EP are each tangent to C_2 and D is inside C_1. If $AB = \frac{5\sqrt2}{2}$, $AD = 2$, $BD = 2$, $EB = 1$, and $ED = \sqrt2$, find AE.

p12. How many ordered pairs (x, y) of positive integers satisfy the following equation?

$$\sqrt{x}+\sqrt{y}=\sqrt{2007}.$$

p13. The sides BC, CA, and CB of triangle ABC have midpoints K, L, and M, respectively. If

$$AB^2 + BC^2 + CA^2 = 200,$$

what is $AK^2 + BL^2 + CM^2$?

p14. Let x and y be real numbers that satisfy:

$$x+\frac{4}{x}=y+\frac{4}{y}=\frac{20}{xy}.$$

Compute the maximum value of $|x-y|$.

p15. 30 math meet teams receive different scores which are then shuffled around to lend an aura of mystery to the grading. What is the probability that no team receives their own score? Express your answer as a decimal accurate to the nearest hundredth.

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