www.artofproblemsolving.com/community/c3168306
by parmenides51

– Team Round

– **p1.** $ABCD$ is a convex quadrilateral such that $AB = 20$, $BC = 24$, $CD = 7$, $DA = 15$, and $\angle DAB$ is a right angle. What is the area of $ABCD$?

p2. A triangular number is one that can be written in the form $1 + 2 + ... + n$ for some positive number n. 1 is clearly both triangular and square. What is the next largest number that is both triangular and square?

p3. Find the last (i.e. rightmost) three digits of 9^{2008}.

p4. When expressing numbers in a base $b \ge 11$, you use letters to represent digits greater than 9. For example, A represents 10 and B represents 11, so that the number 110 in base 10 is $A0$ in base 11. What is the smallest positive integer that has four digits when written in base 10, has at least one letter in its base 12 representation, and no letters in its base 16 representation?

p5. A fly starts from the point $(0, 16)$, then flies straight to the point $(8, 0)$, then straight to the point $(0,-4)$, then straight to the point $(-2, 0)$, and so on, spiraling to the origin, each time intersecting the coordinate axes at a point half as far from the origin as its previous intercept. If the fly flies at a constant speed of 2 units per second, how many seconds will it take the fly to reach the origin?

p6. A line segment is divided into two unequal lengths so that the ratio of the length of the short part to the length of the long part is the same as the ratio of the length of the long part to the length of the whole line segment. Let D be this ratio. Compute

$$D^{-1} + D^{[D^{-1}+D^{(D^{-1}+D^2)}]}.$$

p7. Let $f(x) = 4x + 2$. Find the ordered pair of integers (P,Q) such that their greatest common divisor is 1, P is positive, and for any two real numbers a and b, the sentence:

"$Pa + Qb \ge 0$"

is true if and only if the following sentence is true:

"For all real numbers x, if $|f(x) - 6| < b$, then $|x - 1| < a$."

p8. Call a rectangle "simple" if all four of its vertices have integers as both of their coordinates and has one vertex at the origin. How many simple rectangles are there whose area is less than or equal to 6?

p9. A square is divided into eight congruent triangles by the diagonals and the perpendicular bisectors of its sides. How many ways are there to color the triangles red and blue if two ways that are reflections or rotations of each other are considered the same?

p10. In chess, a knight can move by jumping to any square whose center is $\sqrt{5}$ units away from the center of the square that it is currently on. For example, a knight on the square marked by the horse in the diagram below can move to any of the squares marked with an "X" and to no other squares. How many ways can a knight on the square marked by the horse in the diagram move to the square with a circle in exactly four moves?
https://cdn.artofproblemsolving.com/attachments/d/9/2ef9939642362182af12089f95836d4e29472 png

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Tiebreaker Rpund

– **p1.** (See the diagram below.) $ABCD$ is a square. Points G, H, I, and J are chosen in the interior of $ABCD$ so that:
(i) H is on $\overline{AG}$, I is on $\overline{BH}$, J is on $\overline{CI}$, and G is on $\overline{DJ}$
(ii) $\triangle ABH \sim \triangle BCI \sim \triangle CDJ \sim \triangle DAG$ and
(iii) the radii of the inscribed circles of $\triangle ABH$, $\triangle BCI$, $\triangle CDJ$, $\triangle DAK$, and $GHIJ$ are all the same.
What is the ratio of $\overline{AB}$ to $\overline{GH}$?
https://cdn.artofproblemsolving.com/attachments/f/b/47e8b9c1288874bc48462605ecd06ddf0f251 png

p2. The three solutions r_1, r_2, and r_3 of the equation

$$x^3 + x^2 - 2x - 1 = 0$$

can be written in the form $2\cos(k_1\pi)$, $2\cos(k_2\pi)$, and $2\cos(k_3\pi)$ where $0 \le k_1 < k_2 < k_3 \le 1$. What is the ordered triple (k_1, k_2, k_3)?

p3. P is a convex polyhedron, all of whose faces are either triangles or decagons (10-sided polygon), though not necessarily regular. Furthermore, at each vertex of P exactly three faces meet. If P has 20 triangular faces, how many decagonal faces does P have?

p4. P_1 is a parabola whose line of symmetry is parallel to the x-axis, has $(0, 1)$ as its vertex, and passes through $(2, 2)$. P_2 is a parabola whose line of symmetry is parallel to the y-axis, has $(1, 0)$ as its vertex, and passes through $(2, 2)$. Find all four points of intersection between P_1 and P_2.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Round

– **p1.** Joe owns stock. On Monday morning on October 20th, 2008, his stocks were worth $\$250, 000$. The value of his stocks, for each day from Monday to Friday of that week, increased by 10%, increased by 5%, decreased by 5%, decreased by 15%, and decreased by 20%, though not necessarily in that order. Given this information, let A be the largest possible value of his stocks on that Friday evening, and let B be the smallest possible value of his stocks on that Friday evening. What is $A - B$?

p2. What is the smallest positive integer k such that $2k$ is a perfect square and $3k$ is a perfect cube?

p3. Two competitive ducks decide to have a race in the first quadrant of the xy plane. They both start at the origin, and the race ends when one of the ducks reaches the line $y = \frac12$. The first duck follows the graph of $y = \frac{x}{3}$ and the second duck follows the graph of $y = \frac{x}{5}$. If the two ducks move in such a way that their x-coordinates are the same at any time during the race, find the ratio of the speed of the first duck to that of the second duck when the race ends.

p4. There were grammatical errors in this problem as stated during the contest. The problem should have said:
You play a carnival game as follows: The carnival worker has a circular mat of radius 20 cm, and on top of that is a square mat of side length 10 cm, placed so that the centers of the two mats coincide. The carnival worker also has three disks, one each of radius 1 cm, 2 cm, and 3 cm. You start by paying the worker a modest fee of one dollar, then choosing two of the disks, then throwing the two disks onto the mats, one at a time, so that the center of each disk lies on the circular mat. You win a cash prize if the center of the large disk is on the square AND the large disk touches the small disk, otherwise you just lost the game and you get no money. How much is the cash prize if choosing the two disks randomly and then throwing the disks randomly (i.e. with uniform distribution) will, on average, result in you breaking even?

p5. Four boys and four girls arrive at the Highball High School Senior Ball without a date. The principal, seeking to rectify the situation, asks each of the boys to rank the four girls in decreasing order of preference as a prom date and asks each girl to do the same for the four boys.

None of the boys know any of the girls and vice-versa (otherwise they would have probably found each other before the prom), so all eight teenagers write their rankings randomly. Because the principal lacks the mathematical chops to pair the teenagers together according to their stated preference, he promptly ignores all eight of the lists and randomly pairs each of the boys with a girl. What is the probability that no boy ends up with his third or his fourth choice, and no girl ends up with her third or fourth choice?

p6. In the diagram below, $ABCDEFGH$ is a rectangular prism, $\angle BAF = 30^o$ and $\angle DAH = 60^o$. What is the cosine of $\angle CEG$?
https://cdn.artofproblemsolving.com/attachments/a/1/1af1a7d5d523884703b9ff95aaf301bcc1814
png

p7. Two cows play a game where each has one playing piece, they begin by having the two pieces on opposite vertices of an octahedron, and the two cows take turns moving their piece to an adjacent vertex. The winner is the first player who moves its piece to the vertex occupied by its opponent's piece. Because cows are not the most intelligent of creatures, they move their pieces randomly. What is the probability that the first cow to move eventually wins?

p8. Find the last two digits of
$$\sum_{k=1}^{2008} k\binom{2008}{k}.$$

PS. You had better use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Devil Round

– **p1.** Twelve people, three of whom are in the Mafia and one of whom is a police inspector, randomly sit around a circular table. What is the probability that the inspector ends up sitting next to at least one of the Mafia?

p2. Of the positive integers between 1 and 1000, inclusive, how many of them contain neither the digit "4" nor the digit "7"?

p3. You are really bored one day and decide to invent a variation of chess. In your variation, you create a new piece called the "krook," which, on any given turn, can move either one square up or down, or one square left or right. If you have a krook at the bottom-left corner of the chessboard, how many different ways can the krook reach the top-right corner of the chessboard in exactly 17 moves?

p4. Let p be a prime number. What is the smallest positive integer that has exactly p different positive integer divisors? Write your answer as a formula in terms of p.

p5. You make the square $\{(x, y)| - 5 \le x \le 5, -5 \le y \le 5\}$ into a dartboard as follows:
(i) If a player throws a dart and its distance from the origin is less than one unit, then the player gets 10 points.
(ii) If a player throws a dart and its distance from the origin is between one and three units, inclusive, then the player gets awarded a number of points equal to the number of the quadrant that the dart landed on. (The player receives no points for a dart that lands on the coordinate axes in this case.)
(iii) If a player throws a dart and its distance from the origin is greater than three units, then the player gets 0 points.
If a person throws three darts and each hits the board randomly (i.e with uniform distribution), what is the expected value of the score that they will receive?

p6. Teddy works at Please Forget Meat, a contemporary vegetarian pizza chain in the city of Gridtown, as a deliveryman. Please Forget Meat (PFM) has two convenient locations, marked with "X" and "Y " on the street map of Gridtown shown below. Teddy, who is currently at X, needs to deliver an eggplant pizza to ∇ en route to Y , where he is urgently needed. There is currently construction taking place at A, B, and C, so those three intersections will be completely impassable. How many ways can Teddy get from X to Y while staying on the roads (Traffic tickets are expensive!), not taking paths that are longer than necessary (Gas is expensive!), and that let him pass through ∇ (Losing a job is expensive!)?
`https://cdn.artofproblemsolving.com/attachments/e/0/d4952e923dc97596ad354ed770e80f979740b png`

p7. x, y, and z are positive real numbers that satisfy the following three equations:

$$x + \frac{1}{y} = 4 \quad y + \frac{1}{z} = 1 \quad z + \frac{1}{x} = \frac{7}{3}.$$

Compute xyz.

p8. Alan, Ben, and Catherine will all start working at the Duke University Math Department on January 1st, 2009. Alan's work schedule is on a four-day cycle; he starts by working for three days and then takes one day off. Ben's work schedule is on a seven-day cycle; he starts by working for five days and then takes two days off. Catherine's work schedule is on a ten-day cycle; she starts by working for seven days and then takes three days off. On how many days in 2009 will none of the three be working?

p9. x and y are complex numbers such that $x^3 + y^3 = -16$ and $(x + y)^2 = xy$. What is the value of $|x+y|$?

p10. Call a four-digit number ”well-meaning” if (1) its second digit is the mean of its first and its third digits and (2) its third digit is the mean of its second and fourth digits. How many well-meaning four-digit numbers are there?
(For a four-digit number, its first digit is its thousands [leftmost] digit and its fourth digit is its units [rightmost] digit. Also, four-digit numbers cannot have ”0” as their first digit.)

p11. Suppose that θ is a real number such that $\sum^{\infty} k = 2\sin\left(2^k\theta\right)$ is well-defined and equal to the real number a. Compute:

$$\sum^{\infty} k = 0 \left(\cot^3\left(2^k\theta\right) - \cot\left(2^k\theta\right)\right)\sin^4\left(2^k\theta\right).$$

Write your answer as a formula in terms of a.

p12. You have 13 loaded coins; the probability that they come up as heads are $\cos\left(\frac{0\pi}{24}\right)$,$\cos\left(\frac{1\pi}{24}\right)$, $\cos\left(\frac{2\pi}{24}\right)$, ..., $\cos\left(\frac{11\pi}{24}\right)$ and $\cos\left(\frac{12\pi}{24}\right)$, respectively. You throw all 13 of these coins in the air at once. What is the probability that an even number of them come up as heads?

p13. Three married couples sit down on a long bench together in random order. What is the probability that none of the husbands sit next to their respective wives?

p14. What is the smallest positive integer that has at least 25 different positive divisors?

p15. Let A_1 be any three-element set, $A_2 = \{\emptyset\}$, and $A_3 = \emptyset$. For each $i \in \{1, 2, 3\}$, let:
(i) $B_i = \{\emptyset, A_i\}$,
(ii) C_i be the set of all subsets of B_i,
(iii) $D_i = B_i \cup C_i$, and
(iv) k_i be the number of different elements in D_i.
Compute $k_1k_2k_3$.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).