www.artofproblemsolving.com/community/c3168307
by parmenides51

– Team Round

– **p1.** You are on a flat planet. There are 100 cities at points $x = 1, ..., 100$ along the line $y = -1$, and another 100 cities at points $x = 1, ..., 100$ along the line $y = 1$. The planet's terrain is scalding hot, and you cannot walk over it directly. Instead, you must cross archways from city to city. There are archways between all pairs of cities with different y coordinates, but no other pairs: for instance, there is an archway from $(1,-1)$ to $(50, 1)$, but not from $(1,-1)$ to $(50,-1)$. The amount of "effort" necessary to cross an archway equals the square of the distance between the cities it connects. You are at $(1,-1)$, and you want to get to $(100,-1)$. What is the least amount of effort this journey can take?

p2. Let $f(x) = x^4 + ax^3 + bx^2 + cx + 25$. Suppose a, b, c are integers and $f(x)$ has 4 distinct integer roots. Find $f(3)$.

p3. Frankenstein starts at the point $(0, 0, 0)$ and walks to the point $(3, 3, 3)$. At each step he walks either one unit in the positive x-direction, one unit in the positive y-direction, or one unit in the positive z-direction. How many distinct paths can Frankenstein take to reach his destination?

p4. Let $ABCD$ be a rectangle with $AB = 20$, $BC = 15$. Let X and Y be on the diagonal $\overline{BD}$ of $ABCD$ such that $BX > BY$. Suppose A and X are two vertices of a square which has two sides on lines $\overline{AB}$ and $\overline{AD}$, and suppose that C and Y are vertices of a square which has sides on $\overline{CB}$ and $\overline{CD}$. Find the length XY .
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`png`

p5. $n \ge 2$ kids are trick-or-treating. They enter a haunted house in a single-file line such that each kid is friends with precisely the kids (or kid) adjacent to him. Inside the haunted house, they get mixed up and out of order. They meet up again at the exit, and leave in single file. After leaving, they realize that each kid (except the first to leave) is friends with at least one kid who left before him. In how many possible orders could they have left the haunted house?

p6. Call a set S sparse if every pair of distinct elements of S differ by more than 1. Find the number of sparse subsets (possibly empty) of $\{1, 2, ..., 10\}$.

p7. How many ordered triples of integers (a, b, c) are there such that $1 \le a, b, c \le 70$ and $a^2 +$

$b^2 + c^2$ is divisible by 28?

p8. Let C_1, C_2 be circles with centers O_1, O_2, respectively. Line ℓ is an external tangent to C_1 and C_2, it touches C_1 at A and C_2 at B. Line segment $\overline{O_1O_2}$ meets C_1 at X. Let C be the circle through A, X, B with center O. Let $\overline{OO_1}$ and $\overline{OO_2}$ intersect circle C at D and E, respectively. Suppose the radii of C_1 and C_2 are 16 and 9, respectively, and suppose the area of the quadrilateral O_1O_2BA is 300. Find the length of segment DE.

p9. What is the remainder when $5^{5^{5^5}}$ is divided by 13?

p10. Let α and β be the smallest and largest real numbers satisfying

$$x^2 = 13 + \lfloor x \rfloor + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{4} \right\rfloor .$$

Find $\beta - \alpha$.
($\lfloor a \rfloor$ is defined as the largest integer that is not larger than a.)

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Tiebreaker Round

– **p1**. Your Halloween took a bad turn, and you are trapped on a small rock above a sea of lava. You are on rock 1, and rocks 2 through 12 are arranged in a straight line in front of you. You want to get to rock 12. You must jump from rock to rock, and you can either (1) jump from rock n to $n + 1$ or (2) jump from rock n to $n + 2$. Unfortunately, you are weak from eating too much candy, and you cannot do (2) twice in a row. How many different sequences of jumps will take you to your destination?

p2. Find the number of ordered triples $(p; q; r)$ such that p, q, r are prime, $pq + pr$ is a perfect square and $p + q + r \le 100$.

p3. Let x, y, z be nonzero complex numbers such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ne 0$ and

$$x^2(y + z) + y^2(z + x) + z^2(x + y) = 4(xy + yz + zx) = -3xyz.$$

Find $\frac{x^3+y^3+z^3}{x^2+y^2+z^2}$.

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– Individual Round

– **p1.** Let $p > 5$ be a prime. It is known that the average of all of the prime numbers that are at least 5 and at most p is 12. Find p.

p2. The numbers $1, 2, ..., n$ are written down in random order. What is the probability that $n - 1$ and n are written next to each other? (Give your answer in term of n.)

p3. The Duke Blue Devils are playing a basketball game at home against the UNC Tar Heels. The Tar Heels score N points and the Blue Devils score M points, where $1 < M, N < 100$. The first digit of N is a and the second digit of N is b. It is known that $N = a + b^2$. The first digit of M is b and the second digit of M is a. By how many points do the Blue Devils win?

p4. Let $P(x)$ be a polynomial with integer coefficients. It is known that $P(x)$ gives a remainder of 1 upon polynomial division by $x + 1$ and a remainder of 2 upon polynomial division by $x + 2$. Find the remainder when $P(x)$ is divided by $(x + 1)(x + 2)$.

p5. Dracula starts at the point $(0, 9)$ in the plane. Dracula has to pick up buckets of blood from three rivers, in the following order: the Red River, which is the line $y = 10$; the Maroon River, which is the line $y = 0$; and the Slightly Crimson River, which is the line $x = 10$. After visiting all three rivers, Dracula must then bring the buckets of blood to a castle located at $(8, 5)$. What is the shortest distance that Dracula can walk to accomplish this goal?

p6. Thirteen hungry zombies are sitting at a circular table at a restaurant. They have five identical plates of zombie food. Each plate is either in front of a zombie or between two zombies. If a plate is in front of a zombie, that zombie and both of its neighbors can reach the plate. If a plate is between two zombies, only those two zombies may reach it. In how many ways can we arrange the plates of food around the circle so that each zombie can reach exactly one plate of food? (All zombies are distinct.)

p7. Let R_I , R_{II} ,R_{III} ,R_{IV} be areas of the elliptical region

$$\frac{(x-10)^2}{10}+\frac{(y-31)^2}{31}\le 2009$$

that lie in the first, second, third, and fourth quadrants, respectively. Find $R_I - R_{II} + R_{III} - R_{IV}$.

p8. Let r_1, r_2, r_3 be the three (not necessarily distinct) solutions to the equation $x^3+4x^2-ax+1 = 0$. If a can be any real number, find the minimum possible value of

$$\left(r_1+\frac{1}{r_1}\right)^2+\left(r_2+\frac{1}{r_2}\right)^2+\left(r_3+\frac{1}{r_3}\right)^2$$

p9. Let n be a positive integer. There exist positive integers $1 = a_1 < a_2 < ... < a_n = 2009$ such that the average of any $n-1$ of elements of $\{a_1, a_2, ..., a_n\}$ is a positive integer. Find the maximum possible value of n.

p10. Let $A(0) = (2, 7, 8)$ be an ordered triple. For each n, construct $A(n)$ from $A(n-1)$ by replacing the kth position in $A(n-1)$ by the average (arithmetic mean) of all entries in $A(n-1)$, where $k \equiv n \pmod 3$ and $1 \le k \le 3$. For example, $A(1) = \left(\frac{17}{3}, 7, 8\right)$ and $A(2) = \left(\frac{17}{3}, \frac{62}{9}, 8\right)$. It is known that all entries converge to the same number N. Find the value of N.

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– Devil Round

– **p1.** Find all positive integers n such that $n^3 - 14n^2 + 64n - 93$ is prime.

p2. Let a, b, c be real numbers such that $0 \le a, b, c \le 1$. Find the maximum value of

$$\frac{a}{1+bc}+\frac{b}{1+ac}+\frac{c}{1+ab}$$

p3. Find the maximum value of the function $f(x, y, z) = 4x + 3y + 2z$ on the ellipsoid $16x^2 + 9y^2 + 4z^2 = 1$

p4. Let $x_1, ..., x_n$ be numbers such that $x_1+...+x_n = 2009$. Find the minimum value of $x_1^2+...+x_n^2$ (in term of n).

p5. Find the number of odd integers between 1000 and 9999 that have at least 3 distinct digits.

p6. Let $A_1, A_2, ..., A_{2^n-1}$ be all the possible nonempty subsets of $\{1, 2, 3, ..., n\}$. Find the maximum value of $a_1 + a_2 + ... + a_{2^n-1}$ where $a_i \in A_i$ for each $i = 1, 2, ..., 2^n - 1$.

p7. Find the rightmost digit when 41^{2009} is written in base 7.

p8. How many integral ordered triples (x, y, z) satisfy the equation $x + y + z = 2009$, where $10 \le x < 31$, $100 < z < 310$ and $y \ge 0$.

p9. Scooby has a fair six-sided die, labeled 1 to 6, and Shaggy has a fair twenty-sided die, labeled 1 to 20. During each turn, they both roll their own dice at the same time. They keep rolling the die until one of them rolls a 5. Find the probability that Scooby rolls a 5 before Shaggy does.

p10. Let $N = 1A323492110877$ where A is a digit in the decimal expansion of N. Suppose N is divisible by 7. Find A.

p11. Find all solutions (x, y) of the equation $\tan^4(x + y) + \cot^4(x + y) = 1 - 2x - x^2$, where $-\frac{\pi}{2} \le x; y \le \frac{\pi}{2}$

p12. Find the remainder when $\sum_{k=1}^{50} k!(k^2 + k - 1)$ is divided by 1008.

p13. The devil set of a positive integer n, denoted $D(n)$, is defined as follows:
(1) For every positive integer n, $n \in D(n)$.
(2) If n is divisible by m and $m < n$, then for every element $a \in D(m)$, a^3 must be in $D(n)$.
Furthermore, call a set S scary if for any $a, b \in S$, $a < b$ implies that b is divisible by a. What is the least positive integer n such that $D(n)$ is scary and has at least 2009 elements?

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