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by parmenides51

– Team Round

– **p1.** How many primes $p < 100$ satisfy $p = a^2 + b^2$ for some positive integers a and b?

p2. For $a < b < c$, there exists exactly one Pythagorean triple such that $a + b + c = 2000$. Find $a + c - b$.

p3. Five points lie on the surface of a sphere of radius 1 such that the distance between any two points is at least $\sqrt{2}$. Find the maximum volume enclosed by these five points.

p4. $ABCDEF$ is a convex hexagon with $AB = BC = CD = DE = EF = FA = 5$ and $AC = CE = EA = 6$. Find the area of $ABCDEF$.

p5. Joe and Wanda are playing a game of chance. Each player rolls a fair 11-sided die, whose sides are labeled with numbers $1, 2, ..., 11$. Let the result of the Joe's roll be X, and the result of Wanda's roll be Y . Joe wins if XY has remainder 1 when divided by 11, and Wanda wins otherwise. What is the probability that Joe wins?

p6. Vivek picks a number and then plays a game. At each step of the game, he takes the current number and replaces it with a new number according to the following rule: if the current number n is divisible by 3, he replaces n with $\frac{n}{3} + 2$, and otherwise he replaces n with $\lfloor 3 \log_3 n \rfloor$. If he starts with the number 3^{2011}, what number will he have after 2011 steps?

Note that $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

p7. Define a sequence an of positive real numbers with a$_1 = 1$, and

$$a_{n+1} = \frac{4a_n^2 - 1}{-2 + \frac{4a_n^2-1}{-2+\frac{4a_n^2-1}{-2+...}}}.$$

What is a_{2011}?

p8. A set S of positive integers is called good if for any $x, y \in S$ either $x = y$ or $|x - y| \geq 3$. How many subsets of $\{1, 2, 3, ..., 13\}$ are good? Include the empty set in your count.

p9. Find all pairs of positive integers (a, b) with $a \le b$ such that $10 \cdot lcm\,(a, b) = a^2 + b^2$. Note that $lcm\,(m, n)$ denotes the least common multiple of m and n.

p10. For a natural number n, $g(n)$ denotes the largest odd divisor of n. Find

$$g(1) + g(2) + g(3) + ... + g(2^{2011})$$

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Tiebreaker Round

– **p1.** 2011 distinct points are arranged along the perimeter of a circle. We choose without replacement four points P, Q, R, S. What is the probability that no two of the segments PQ, QR, RS, SP intersect (disregarding the endpoints)?

p2. In Soviet Russia, all phone numbers are between three and six digits and contain only the digits 1, 2, and 3. No phone number may be the prefix of another phone number, so, for example, we cannot have the phone numbers 123 and 12332. If the Soviet bureaucracy has preassigned 10 phone numbers of length 3, 20 numbers of length 4, and 77 phone numbers of length 6, what is the maximum number of phone numbers of length 5 that the authorities can allocate?

p3. The sequence $\{a_n\}_{n\ge 1}$ is defined as follows: we have $a_1 = 1$, $a_2 = 0$, and for $n \ge 3$ we have

$$a_n = \frac{1}{2} \sum_{\substack{1\le i,j \\ i+j+k=n}} a_ia_ja_k.$$

Find

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

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– Individual Round

– **p1.** Elsie M. is fixing a watch with three gears. Gear A makes a full rotation every 5 minutes, gear B makes a full rotation every 8 minutes, and gear C makes a full rotation every 12 minutes. The gears continue spinning until all three gears are in their original positions at the same time. How many minutes will it take for the gears to stop spinning?

p2. Optimus has to pick 10 distinct numbers from the set of positive integers $\{2, 3, 4, ..., 29, 30\}$. Denote the numbers he picks by $\{a_1, a_2, ..., a_{10}\}$. What is the least possible value of

$$d(a_1) + d(a_2) + ... + d(a_{10}),$$

where $d(n)$ denotes the number of positive integer divisors of n? For example, $d(33) = 4$ since 1, 3, 11, and 33 divide 33.

p3. Michael is given a large supply of both 1×3 and 1×5 dominoes and is asked to arrange some of them to form a 6×13 rectangle with one corner square removed. What is the minimum number of 1×3 dominoes that Michael can use?
https://cdn.artofproblemsolving.com/attachments/6/6/c6a3ef7325ecee417e37ec9edb5374aceab9f
png

p4. Andy, Ben, and Chime are playing a game. The probabilities that each player wins the game are, respectively, the roots a, b, and c of the polynomial $x^3 - x^2 + \frac{111}{400}x - \frac{9}{400} = 0$ with $a \le b \le c$. If they play the game twice, what is the probability of the same player winning twice?

p5. TongTong is doodling in class and draws a 3×3 grid. She then decides to color some (that is, at least one) of the squares blue, such that no two 1×1 squares that share an edge or a corner are both colored blue. In how many ways may TongTong color some of the squares blue? TongTong cannot rotate or reflect the board.
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p6. Given a positive integer n, we define $f(n)$ to be the smallest possible value of the expression

$$|\square 1 \square 2 ... \square n|,$$

where we may place a $+$ or a $-$ sign in each box. So, for example, $f(3) = 0$, since $|+1+2-3| = 0$. What is $f(1) + f(2) + ... + f(2011)$?

p7. The Duke Men's Basketball team plays 11 home games this season. For each game, the team has a $\frac{3}{4}$ probability of winning, except for the UNC game, which Duke has a $\frac{9}{10}$ probability of winning. What is the probability that Duke wins an odd number of home games this season?

p8. What is the sum of all integers n such that $n^2 + 2n + 2$ divides $n^3 + 4n^2 + 4n - 14$?

p9. Let $\{a_n\}_{n=1}^{N}$ be a finite sequence of increasing positive real numbers with $a_1 < 1$ such that

$$a_{n+1} = a_n\sqrt{1 - a_1^2} + a_1\sqrt{1 - a_n^2}$$

and $a_{10} = 1/2$. What is a_{20}?

p10. Three congruent circles are placed inside a unit square such that they do not overlap. What is the largest
possible radius of one of these circles?

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– Devil Round

– Round 1

p1. The fractal T-shirt for this year's Duke Math Meet is so complicated that the printer broke trying to print it. Thus, we devised a method for manually assembling each shirt - starting with the full-size 'base' shirt, we paste a smaller shirt on top of it. And then we paste an even smaller shirt on top of that one. And so on, infinitely many times. (As you can imagine, it took a while to make all the shirts.) The completed T-shirt consists of the original 'base' shirt along with all of the shirts we pasted onto it. Now suppose the base shirt requires $2011\ cm^2$ of fabric to make, and that each pasted-on shirt requires $4/5$ as much fabric as the previous one did. How many cm^2 of fabric in total are required to make one complete shirt?

p2. A dog is allowed to roam a yard while attached to a 60-meter leash. The leash is anchored to a 40-meter by 20-meter rectangular house at the midpoint of one of the long sides of the house. What is the total area of the yard that the dog can roam?

p3. 10 birds are chirping on a telephone wire. Bird 1 chirps once per second, bird 2 chirps once every 2 seconds, and so on through bird 10, which chirps every 10 seconds. At time $t = 0$, each bird chirps. Define $f(t)$ to be the number of birds that chirp during the t^{th} second. What is the smallest $t > 0$ such that $f(t)$ and $f(t + 1)$ are both at least 4?

Round 2

p4. The answer to this problem is 3 times the answer to problem 5 minus 4 times the answer to problem 6 plus 1.

p5. The answer to this problem is the answer to problem 4 minus 4 times the answer to problem 6 minus 1.

p6. The answer to this problem is the answer to problem 4 minus 2 times the answer to problem 5.

Round 3

p7. Vivek and Daniel are playing a game. The game ends when one person wins 5 rounds. The probability that either wins the first round is $1/2$. In each subsequent round the players have a probability of winning equal to the fraction of games that the player has lost. What is the probability that Vivek wins in six rounds?

p8. What is the coefficient of x^8y^7 in $(1 + x^2 - 3xy + y^2)^{17}$?

p9. Let $U(k)$ be the set of complex numbers z such that $z^k = 1$. How many distinct elements are in the union of $U(1), U(2), ..., U(10)$?

Round 4

p10. Evaluate $29\binom{30}{0}+28\binom{30}{1}+27\binom{30}{2}+...+0\binom{30}{29}-\binom{30}{30}$. You may leave your answer in exponential format.

p11. What is the number of strings consisting of $2a$s, $3b$s and $4c$s such that a is not immediately followed by b, b is not immediately followed by c and c is not immediately followed by a?

p12. Compute $\left(\sqrt{3} + \tan(1^o)\right)\left(\sqrt{3} + \tan(2^o)\right) ... \left(\sqrt{3} + \tan(29^o)\right)$.

Round 5

p13. Three massless legs are randomly nailed to the perimeter of a massive circular wooden table with uniform density. What is the probability that the table will not fall over when it is set on its legs?

p14. Compute

$$\sum_{n=1}^{2011} \frac{n+4}{n(n+1)(n+2)(n+3)}$$

p15. Find a polynomial in two variables with integer coefficients whose range is the positive real numbers.

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