www.artofproblemsolving.com/community/c3168323
by parmenides51

– Team Round

– **p1.** What is the maximum possible value of m such that there exist m integers $a_1, a_2, ..., a_m$ where all the decimal representations of $a_1!, a_2!, ..., a_m!$ end with the same amount of zeros?

p2. Let $f : R \to R$ be a function such that $f(x) + f(y^2) = f(x^2 + y)$, for all $x, y \in R$. Find the sum of all possible $f(-2017)$.

p3. What is the sum of prime factors of 1000027?

p4. Let
$$\frac{1}{2!} + \frac{2}{3!} + ... + \frac{2016}{2017!} = \frac{n}{m},$$
where n, m are relatively prime. Find $(m - n)$.

p5. Determine the number of ordered pairs of real numbers (x, y) such that $\sqrt[3]{3 - x^3 - y^3} = \sqrt{2 - x^2 - y^2}$

p6. Triangle $\triangle ABC$ has $\angle B = 120^o$, $AB = 1$. Find the largest real number x such that $CA - CB > x$ for all possible triangles $\triangle ABC$.

p7. Jung and Remy are playing a game with an unfair coin. The coin has a probability of p where its outcome is heads. Each round, Jung and Remy take turns to flip the coin, starting with Jung in round 1. Whoever gets heads first wins the game. Given that Jung has the probability of $8/15$, what is the value of p?

p8. Consider a circle with 7 equally spaced points marked on it. Each point is 1 unit distance away from its neighbors and labelled $0, 1, 2, ..., 6$ in that order counterclockwise. Feng is to jump around the circle, starting at the point 0 and making six jumps counterclockwise with distinct lengths $a_1, a_2, ..., a_6$ in a way such that he will land on all other six nonzero points afterwards. Let s denote the maximum value of a_i. What is the minimum possible value of s?

p9. Justin has a $4 \times 4 \times 4$ colorless cube that is made of 64 unit-cubes. He then colors m unit-cubes such that none of them belong to the same column or row of the original cube. What is the largest possible value of m?

p10. Yikai wants to know Liang's secret code which is a 6-digit integer x. Furthermore, let $d(n)$ denote the digital sum of a positive integer n. For instance, $d(14) = 5$ and $d(3) = 3$. It is given that

$$x + d(x) + d(d(x)) + d(d(d(x))) = 999868.$$

Please find x.

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Tiebreaker Round

– **p1.** Find the sum of all 3-digit positive integers $\overline{abc}$ that satisfy

$$\overline{abc} = \binom{n}{a} + \binom{n}{b} + \binom{n}{c}$$

for some $n \le 10$.

p2. Feng and Trung play a game. Feng chooses an integer p from 1 to 90, and Trung tries to guess it. In each round, Trung asks Feng two yes-or-no questions about p. Feng must answer one question truthfully and one question untruthfully. After 15 rounds, Trung concludes there are n possible values for p. What is the least possible value of n, assuming Feng chooses the best strategy to prevent Trung from guessing correctly?

p3. A hypercube H_n is an n-dimensional analogue of a cube. Its vertices are all the points $(x_1, .., x_n)$ that satisfy $x_i = 0$ or 1 for all $1 \le i \le n$ and its edges are all segments that connect two adjacent vertices. (Two vertices are adjacent if their coordinates differ at exactly one x_i . For example, $(0, 0, 0, 0)$ and $(0, 0, 0, 1)$ are adjacent on H_4.) Let $\phi(H_n)$ be the number of cubes formed by the edges and vertices of H_n. Find $\phi(H_4) + \phi(H_5)$.

p4. Denote the legs of a right triangle as a and b, the radius of the circumscribed circle as R and the radius of the inscribed circle as r. Find $\frac{a+b}{R+r}$.

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– Individual Round

– **p1.** How many subsets of $\{D, U, K, E\}$ have an odd number of elements?

p2. Find the coefficient of x^{12} in $(1 + x^2 + x^4 + ... + x^{28})(1 + x + x^2 + ... + x^{14})^2$.

p3. How many 4-digit numbers have their digits in non-decreasing order from left to right?

p4. A dodecahedron (a polyhedron with 12 faces, each a regular pentagon) is projected orthogonally onto a plane parallel to one of its faces to form a polygon. Find the measure (in degrees) of the largest interior angle of this polygon.

p5. Justin is back with a 6×6 grid made of 36 colorless squares. Dr. Kraines wants him to color some squares such that • Each row and column of the grid must have at least one colored square • For each colored square, there must be another colored square on the same row or column
What is the minimum number of squares that Justin will have to color?

p6. Inside a circle C, we have three equal circles C_1, C_2, C_3, which are pairwise externally tangent to each other and all internally tangent to C. What is the ratio of the area of C_1 to the area of C?

p7. There are 3 different paths between the Duke Chapel and the Physics building. 6 students are heading towards the Physics building for a class, so they split into 3 pairs and each pair takes a separate path from the Chapel. After class, they again split into 3 pairs and take separate paths back. Find the number of possible scenarios where each student's companion on the way there is different from their companion on the way back.

p8. Let a_n be a sequence that satisfies the recurrence relation

$$a_n a_{n+2} = \frac{\cos(3a_{n+1})}{\cos(a_{n+1})[2\cos(2a_{n+1}) - 1]} a_{n+1}$$

with $a_1 = 2$ and $a_2 = 3$. Find the value of $2018a_{2017}$.

p9. Let $f(x)$ be a polynomial with minimum degree, integer coefficients, and leading coefficient of 1 that satisfies $f(\sqrt{7} + \sqrt{13}) = 0$. What is the value of $f(10)$?

p10. 1024 Duke students, indexed 1 to 1024, are having a chat. For each $1 \le i \le 1023$, student i claims that student $2^{\lfloor \log_2 i \rfloor + 1}$ has a girlfriend. ($\lfloor x \rfloor$ is the greatest integer less than or equal to

x.) Given that exactly 201 people are lying, find the index of the 61st liar (ordered by index from smallest to largest).

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– Devil Round

– **p1.** Let $A = \{D, U, K, E\}$ and $B = \{M, A, T, H\}$. How many maps are there from A to B?

p2. The product of two positive integers x and y is equal to 3 more than their sum. Find the sum of all possible x.

p3. There is a bag with 1 red ball and 1 blue ball. Jung takes out a ball at random and replaces it with a red ball. Remy then draws a ball at random. Given that Remy drew a red ball, what is the probability that the ball Jung took was red?

p4. Let $ABCDE$ be a regular pentagon and let AD intersect BE at P. Find $\angle APB$.

p5. It is Justin and his $4 \times 4 \times 4$ cube again! Now he uses many colors to color all unit-cubes in a way such that two cubes on the same row or column must have different colors. What is the minimum number of colors that Justin needs in order to do so?

p6. $f(x)$ is a polynomial of degree 3 where $f(1) = f(2) = f(3) = 4$ and $f(-1) = 52$. Determine $f(0)$.

p7. Mike and Cassie are partners for the Duke Problem Solving Team and they decide to meet between 1 pm and 2 pm. The one who arrives first will wait for the other for 10 minutes, the lave. Assume they arrive at any time between 1 pm and 2 pm with uniform probability. Find the probability they meet.

p8. The remainder of $2x^3 - 6x^2 + 3x + 5$ divided by $(x - 2)^2$ has the form $ax + b$. Find ab.

p9. Find m such that the decimal representation of m! ends with exactly 99 zeros.

p10. Let $1000 \le n = \overline{DUKE} \le 9999$. be a positive integer whose digits $\overline{DUKE}$ satisfy the divisibility condition:

$$1111 | \left(\overline{DUKE} + \overline{DU} \times \overline{KE}\right)$$

Determine the smallest possible value of n.

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