www.artofproblemsolving.com/community/c3168325
by parmenides51

– Team Round

– **p1.** Zion, RJ, Cam, and Tre decide to start learning languages. The four most popular languages that Duke offers are Spanish, French, Latin, and Korean. If each friend wants to learn exactly three of these four languages, how many ways can they pick courses such that they all attend at least one course together?

p2. Suppose we wrote the integers between 0001 and 2019 on a blackboard as such:

$$00010002000320182019.$$

How many 0's did we write?

p3. Duke's basketball team has made x three-pointers, y two-pointers, and z one-point free throws, where x, y, z are whole numbers. Given that $3|x$, $5|y$, and $7|z$, find the greatest number of points that Duke's basketball team could not have scored.

p4. Find the minimum value of $x^2+2xy+3y^2+4x+8y+12$, given that x and y are real numbers.

Note: calculus is not required to solve this problem.

p5. Circles C_1, C_2 have radii $1, 2$ and are centered at O_1, O_2, respectively. They intersect at points A and B, and convex quadrilateral O_1AO_2B is cyclic. Find the length of AB. Express your answer as $x/\sqrt{y}$, where x, y are integers and y is square-free.

p6. An infinite geometric sequence $\{a_n\}$ has sum $\sum_{n=0}^{\infty} a_n = 3$. Compute the maximum possible value of the sum $\sum_{n=0}^{\infty} a_{3n}$.

p7. Let there be a sequence of numbers $x_1, x_2, x_3, ...$ such that for all i,

$$x_i = \frac{49}{7^{\frac{i}{1010}}+49}.$$

Find the largest value of n such that

$$\left\lfloor \sum_{i=1} nx_i \right\rfloor \le 2019.$$

p8. Let X be a 9-digit integer that includes all the digits 1 through 9 exactly once, such that any 2-digit number formed from adjacent digits of X is divisible by 7 or 13. Find all possible values of X.

p9. Two 2025-digit numbers, $428\underbrace{99...99}_{2019\ \ 9\text{'s}}571$ and $571\underbrace{99...99}_{2019\ \ 9\text{'s}}428$, form the legs of a right triangle. Find the sum of the digits in the hypotenuse.

p10. Suppose that the side lengths of $\triangle ABC$ are positive integers and the perimeter of the triangle is 35. Let G the centroid and I be the incenter of the triangle. Given that $\angle GIC = 90^o$, what is the length of AB?

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Tiebreaker Round

– **p1.** Let $a(1), a(2), ..., a(n), ...$ be an increasing sequence of positive integers satisfying $a(a(n)) = 3n$ for every positive integer n. Compute $a(2019)$.

p2. Consider the function $f(12x-7) = 18x^3 - 5x + 1$. Then, $f(x)$ can be expressed as $f(x) = ax^3 + bx^2 + cx + d$, for some real numbers a, b, c and d. Find the value of $(a + c)(b + d)$.

p3. Let a, b be real numbers such that $\sqrt{5+2\sqrt{6}} = \sqrt{a} + \sqrt{b}$. Find the largest value of the quantity

$$X = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + ...}}}$$

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– Individual Round

– **p1.** Compute the value of N, where

$$N = 818^3 - 6 \cdot 818^2 \cdot 209 + 12 \cdot 818 \cdot 209^2 - 8 \cdot 209^3$$

p2. Suppose $x \le 2019$ is a positive integer that is divisible by 2 and 5, but not 3. If 7 is one of the digits in x, how many possible values of x are there?

p3. Find all non-negative integer solutions (a, b) to the equation
$$b^2 + b + 1 = a^2.$$

p4. Compute the remainder when $\sum_{n=1}^{2019} n^4$ is divided by 53.

p5. Let ABC be an equilateral triangle and $CDEF$ a square such that E lies on segment AB and F on segment BC. If the perimeter of the square is equal to 4, what is the area of triangle ABC?
`https://cdn.artofproblemsolving.com/attachments/1/6/52d9ef7032c2fadd4f97d7c0ea051b3766b58`
`png`

p6.
$$S = \frac{4}{1\times 2\times 3} + \frac{5}{2\times 3\times 4} + \frac{6}{3\times 4\times 5} + ... + \frac{101}{98\times 99\times 100}$$
Let $T = \frac{5}{4} - S$. If $T = \frac{m}{n}$, where m and n are relatively prime integers, find the value of $m + n$.

p7. Find the sum of
$$\sum_{i=0}^{2019} \frac{2^i}{2^i + 2^{2019-i}}$$

p8. Let A and B be two points in the Cartesian plane such that A lies on the line $y = 12$, and B lies on the line $y = 3$. Let C_1, C_2 be two distinct circles that intersect both A and B and are tangent to the x-axis at P and Q, respectively. If $PQ = 420$, determine the length of AB.

p9. Zion has an average 2 out of 3 hit rate for 2-pointers and 1 out of 3 hit rate for 3-pointers. In a recent basketball match, Zion scored 18 points without missing a shot, and all the points came from 2 or 3-pointers. What is the probability that all his shots were 3-pointers?

p10. Let $S = \{1, 2, 3, ..., 2019\}$. Find the number of non-constant functions $f : S \to S$ such that
$$f(k) = f(f(k + 1)) \le f(k + 1) \quad for \quad all \quad 1 \le k \le 2018.$$
Express your answer in the form $\binom{m}{n}$, where m and n are integers.

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