www.artofproblemsolving.com/community/c3168326
by parmenides51

– Team Round

– **p1.** At Duke, $1/2$ of the students like lacrosse, $3/4$ like football, and $7/8$ like basketball. Let p be the proportion of students who like at least all three of these sports and let q be the difference between the maximum and minimum possible values of p. If q is written as m/n in lowest terms, find the value of $m+n$.

p2. A *dukie* word is a 10-letter word, each letter is one of the four D, U, K, E such that there are four consecutive letters in that word forming the letter $DUKE$ in this order. For example, $DUDKDUKEEK$ is a dukie word, but $DUEDKUKEDE$ is not. How many different dukie words can we construct in total?

p3. Rectangle $ABCD$ has sides $AB = 8$, $BC = 6$. $\triangle$ AEC is an isosceles right triangle with hypotenuse AC and E above AC. $\triangle$ BFD is an isosceles right triangle with hypotenuse BD and F below BD. Find the area of $BCFE$.

p4. Chris is playing with 6 pumpkins. He decides to cut each pumpkin in half horizontally into a top half and a bottom half. He then pairs each top-half pumpkin with a bottom-half pumpkin, so that he ends up having six ”recombinant pumpkins”. In how many ways can he pair them so that only one of the six top-half pumpkins is paired with its original bottom-half pumpkin?

p5. Matt comes to a pumpkin farm to pick 3 pumpkins. He picks the pumpkins randomly from a total of 30 pumpkins. Every pumpkin weighs an integer value between 7 to 16 (including 7 and 16) pounds, and there're 3 pumpkins for each integer weight between 7 to 16. Matt hopes the weight of the 3 pumpkins he picks to form the length of the sides of a triangle. Let m/n be the probability, in lowest terms, that Matt will get what he hopes for. Find the value of $m+n$

p6. Let a, b, c, d be distinct complex numbers such that $|a| = |b| = |c| = |d| = 3$ and $|a+b+c+d| = 8$. Find $|abc + abd + acd + bcd|$.

p7. A board contains the integers $1, 2, ..., 10$. Anna repeatedly erases two numbers a and b and replaces it with $a+b$, gaining $ab(a+b)$ lollipops in the process. She stops when there is only one number left in the board. Assuming Anna uses the best strategy to get the maximum number of lollipops, how many lollipops will she have?

p8. Ajay and Joey are playing a card game. Ajay has cards labelled $2, 4, 6, 8$, and 10, and Joey has cards labelled $1, 3, 5, 7, 9$. Each of them takes a hand of 4 random cards and picks one to play. If one of the cards is at least twice as big as the other, whoever played the smaller card wins. Otherwise, the larger card wins. Ajay and Joey have big brains, so they play perfectly. If m/n is the probability, in lowest terms, that Joey wins, find $m + n$.

p9. Let $ABCDEFGHI$ be a regular nonagon with circumcircle ω and center O. Let M be the midpoint of the shorter arc AB of ω, P be the midpoint of MO, and N be the midpoint of BC. Let lines OC and PN intersect at Q. Find the measure of $\angle NQC$ in degrees.

p10. In a 30×30 square table, every square contains either a kit-kat or an oreo. Let T be the number of triples (s_1, s_2, s_3) of squares such that s_1 and s_2 are in the same row, and s_2 and s_3 are in the same column, with s_1 and s_3 containing kit-kats and s_2 containing an oreo. Find the maximum value of T.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Round

– **p1.** Four witches are riding their brooms around a circle with circumference 10 m. They are standing at the same spot, and then they all start to ride clockwise with the speed of 1, 2, 3, and 4 m/s, respectively. Assume that they stop at the time when every pair of witches has met for at least two times (the first position before they start counts as one time). What is the total distance all the four witches have travelled?

p2. Suppose A is an equilateral triangle, O is its inscribed circle, and B is another equilateral triangle inscribed in O. Denote the area of triangle T as $[T]$. Evaluate $\frac{[A]}{[B]}$.

p3. Tim has bought a lot of candies for Halloween, but unfortunately, he forgot the exact number of candies he has. He only remembers that it's an even number less than 2020. As Tim tries to put the candies into his unlimited supply of boxes, he finds that there will be 1 candy left if he puts seven in each box, 6 left if he puts eleven in each box, and 3 left if he puts thirteen in each box. Given the above information, find the total number of candies Tim has bought.

p4. Let $f(n)$ be a function defined on positive integers n such that $f(1) = 0$, and $f(p) = 1$ for all prime numbers p, and

$$f(mn) = nf(m) + mf(n)$$

for all positive integers m and n. Let

$$n = 277945762500 = 2^2 3^3 5^5 7^7$$

Compute the value of $\frac{f(n)}{n}$.

p5. Compute the only positive integer value of $\frac{404}{r^2-4}$, where r is a rational number.

p6. Let $a = 3 + \sqrt{10}$. If

$$\prod_{k=1}^{\infty} \left(1 + \frac{5a+1}{a^k + a}\right) = m + \sqrt{n},$$

where m and n are integers, find $10m + n$.

p7. Charlie is watching a spider in the center of a hexagonal web of side length 4. The web also consists of threads that form equilateral triangles of side length 1 that perfectly tile the hexagon. Each minute, the spider moves unit distance along one thread. If $\frac{m}{n}$ is the probability, in lowest terms, that after four minutes the spider is either at the edge of her web or in the center, find the value of $m + n$.

p8. Let ABC be a triangle with $AB = 10$; $AC = 12$, and ω its circumcircle. Let F and G be points on $\overline{AC}$ such that $AF = 2$, $FG = 6$, and $GC = 4$, and let $\overrightarrow{BF}$ and $\overrightarrow{BG}$ intersect ω at D and E, respectively. Given that AC and DE are parallel, what is the square of the length of BC?

p9. Two blue devils and 4 angels go trick-or-treating. They randomly split up into 3 non-empty groups. Let p be the probability that in at least one of these groups, the number of angels is nonzero and no more than the number of devils in that group. If $p = \frac{m}{n}$ in lowest terms, compute $m + n$.

p10. We know that

$$2^{22000} = \underbrace{4569878...229376}_{6623\ digits}.$$

For how many positive integers $n < 22000$ is it also true that the first digit of 2^n is 4?

PS. You had better use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).