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– Team Round

– **p1.** In basketball, teams can score $1, 2$, or 3 points each time. Suppose that Duke basketball have scored 8 points so far. What is the total number of possible ways (ordered) that they have scored?
For example, $(1, 2, 2, 2, 1)$,$(1, 1, 2, 2, 2)$ are two different ways.

p2. All the positive integers that are coprime to 2021 are grouped in increasing order, such that the nth group contains $2n - 1$ numbers. Hence the first three groups are $\{1\}$, $\{2, 3, 4\}$, $\{5, 6, 7, 8, 9\}$. Suppose that 2022 belongs to the kth group. Find k.

p3. Let $A = (0, 0)$ and $B = (3, 0)$ be points in the Cartesian plane. If R is the set of all points X such that $\angle AXB \ge 60^o$ (all angles are between 0^o and 180^o), find the integer that is closest to the area of R.

p4. What is the smallest positive integer greater than 9 such that when its left-most digit is erased, the resulting number is one twenty-ninth of the original number?

p5. Jonathan is operating a projector in the cartesian plane. He sets up 2 infinitely long mirrors represented by the lines $y = \tan(15^o)x$ and $y = 0$, and he places the projector at $(1, 0)$ pointed perpendicularly to the x-axis in the positive y direction. Jonathan furthermore places a screen on one of the mirrors such that light from the projector reflects off the mirrors a total of three times before hitting the screen. Suppose that the coordinates of the screen is (a, b). Find $10a^2 + 5b^2$.

p6. Dr Kraines has a cube of size $5 \times 5 \times 5$, which is made from 5^3 unit cubes. He then decides to choose m unit cubes that have an outside face such that any two different cubes don't share a common vertex. What is the maximum value of m?

p7. Let $a_n = \tan^{-1}(n)$ for all positive integers n. Suppose that

$$\sum_{k=4}^{\infty}(-1)^{\lfloor \frac{k}{2} \rfloor +1} \tan(2a_k)$$

is equals to a/b , where a, b are relatively prime. Find $a + b$.

p8. Rishabh needs to settle some debts. He owes 90 people and he must pay \$ $(101050 + n)$ to the nth person where $1 \le n \le 90$. Rishabh can withdraw from his account as many coins of values \$ 2021 and \$ x for some fixed positive integer x as is necessary to pay these debts. Find the sum of the four least values of x so that there exists a person to whom Rishabh is unable to pay the exact amount owed using coins.

p9. A frog starts at $(1, 1)$. Every second, if the frog is at point (x, y), it moves to $(x + 1, y)$ with probability $\frac{x}{x+y}$ and moves to $(x, y + 1)$ with probability $\frac{y}{x+y}$. The frog stops moving when its y coordinate is 10. Suppose the probability that when the frog stops its x-coordinate is strictly less than 16, is given by m/n where m, n are positive integers that are relatively prime. Find $m + n$.

p10. In the triangle ABC, $AB = 585$, $BC = 520$, $CA = 455$. Define X, Y to be points on the segment BC. Let $Z \ne A$ be the intersection of AY with the circumcircle of ABC. Suppose that XZ is parallel to AC and the circumcircle of XYZ is tangent to the circumcircle of ABC at Z. Find the length of XY .

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Tiebreaker Round

– You are standing on one of the faces of a cube. Each turn, you randomly choose another face that shares an edge with the face you are standing on with equal probability, and move to that face. Let $F(n)$ the probability that you land on the starting face after n turns. Supposed that $F(8) = \frac{43}{256}$, and F(10) can be expressed as a simplified fraction $\frac{a}{b}$. Find $a + b$.

– Individual Round

– **p1.** There are 4 mirrors facing the inside of a 5×7 rectangle as shown in the figure. A ray of light comes into the inside of a rectangle through A with an angle of 45^o. When it hits the sides of the rectangle, it bounces off at the same angle, as shown in the diagram. How many times will the ray of light bounce before it reaches any one of the corners A, B, C, D? A bounce is a time when the ray hit a mirror and reflects off it.
`https://cdn.artofproblemsolving.com/attachments/1/e/d6ea83941cdb4b2dab187d09a0c45782af169`
`png`

p2. Jerry cuts 4 unit squares out from the corners of a 45×45 square and folds it into a $43 \times 43 \times 1$ tray. He then divides the bottom of the tray into a 43×43 grid and drops a unit cube, which lands in precisely one of the squares on the grid with uniform probability. Suppose that the average number of sides of the cube that are in contact with the tray is given by $\frac{m}{n}$ where m, n are positive integers that are relatively prime. Find $m + n$.

p3. Compute $2021^4 - 4 \cdot 2023^4 + 6 \cdot 2025^4 - 4 \cdot 2027^4 + 2029^4$.

p4. Find the number of distinct subsets $S \subseteq \{1, 2, ..., 20\}$, such that the sum of elements in S leaves a remainder of 10 when divided by 32.

p5. Some k consecutive integers have the sum 45. What is the maximum value of k?

p6. Jerry picks 4 distinct diagonals from a regular nonagon (a regular polygon with 9-sides). A diagonal is a segment connecting two vertices of the nonagon that is not a side. Let the probability that no two of these diagonals are parallel be $\frac{m}{n}$ where m, n are positive integers that are relatively prime. Find $m + n$.

p7. The Olympic logo is made of 5 circles of radius 1, as shown in the figure `https://cdn.artofproblemsolving.com/attachments/1/7/9dafe6b72aa8471234afbaf4c51e3e97c49ee5.png` Suppose that the total area covered by these 5 circles is $a + b\pi$ where a, b are rational numbers. Find $10a + 20b$.

p8. Let $P(x)$ be an integer polynomial (polynomial with integer coefficients) with $P(-5) = 3$ and $P(5) = 23$. Find the minimum possible value of $|P(-2) + P(2)|$.

p9. There exists a unique tuple of rational numbers (a, b, c) such that the equation

$$a \log 10 + b \log 12 + c \log 90 = \log 2025.$$

What is the value of $a + b + c$?

p10. Each grid of a board 7×7 is filled with a natural number smaller than 7 such that the number in the grid at the ith row and jth column is congruent to $i + j$ modulo 7. Now, we can choose any two different columns or two different rows, and swap them. How many different boards can we obtain from a finite number of swaps?

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