Math Prize for Girls 2022 Problems

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by Ravi B

1 Determine the real value of t that minimizes the expression

$$\sqrt{t^2 + (t^2 - 1)^2} + \sqrt{(t - 14)^2 + (t^2 - 46)^2}.$$

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2 Let b and c be random integers from the set $\{1, 2, \ldots, 100\}$, chosen uniformly and independently. What is the probability that the roots of the quadratic $x^2 + bx + c$ are real?

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3 Let $ABCD$ be a square face of a cube with edge length 2. A plane P that contains A and the midpoint of $\overline{BC}$ splits the cube into two pieces of the same volume. What is the square of the area of the intersection of P and the cube?

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4 Determine the largest integer n such that $n < 103$ and $n^3 - 1$ is divisible by 103.

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5 Given a real number a, the *floor* of a, written $\lfloor a \rfloor$, is the greatest integer less than or equal to a. For how many real numbers x such that $1 \le x \le 20$ is

$$x^2 + \lfloor 2x \rfloor = \lfloor x^2 \rfloor + 2x\,?$$

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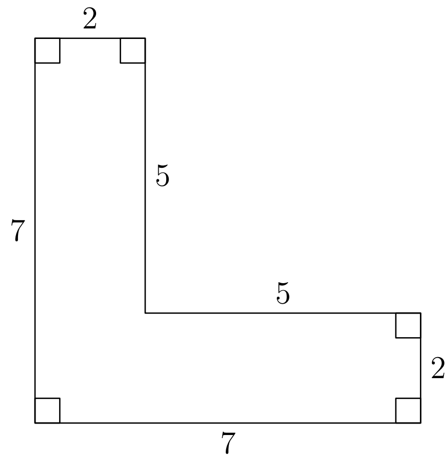
6 An L-shaped region is formed by attaching two 2 by 5 rectangles to adjacent sides of a 2 by 2 square as shown below.



The resulting shape has an area of 24 square units. How many ways are there to tile this shape with 2 by 1 dominos (each of which may be placed horizontally or vertically)?

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7 The quadrilateral $ABCD$ is an isosceles trapezoid with $AB = CD = 1$, $BC = 2$, and $DA = 1 + \sqrt{3}$. What is the measure of $\angle ACD$ in degrees?

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8 Let S be the set of numbers of the form $n^5 - 5n^3 + 4n$, where n is an integer that is not a multiple of 3. What is the largest integer that is a divisor of every number in S?

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9 Let $\triangle PQO$ be the unique right isosceles triangle inscribed in the parabola $y = 12x^2$ with P in the first quadrant, right angle at Q in the second quadrant, and O at the vertex $(0, 0)$. Let $\triangle ABV$ be the unique right isosceles triangle inscribed in the parabola $y = x^2/5 + 1$ with A in the first quadrant, right angle at B in the second quadrant, and V at the vertex $(0, 1)$. The y-coordinate of A can be uniquely written as $uq^2 + vq + w$, where q is the x-coordinate of Q and u, v, and w are integers. Determine $u + v + w$.

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10 An algal cell population is found to have a_k cells on day k. Each day, the number of cells at least doubles. If $a_0 \geq 1$ and $a_3 \leq 60$, how many quadruples of integers (a_0, a_1, a_2, a_3) could represent the algal cell population size on the first 4 days?

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11 Let A, B, C, D, E, and F be 6 points around a circle, listed in clockwise order. We have $AB = 3\sqrt{2}$, $BC = 3\sqrt{3}$, $CD = 6\sqrt{6}$, $DE = 4\sqrt{2}$, and $EF = 5\sqrt{2}$. Given that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent, determine the square of AF.

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12 Solve the equation

$$\sin 9^\circ \sin 21^\circ \sin(102^\circ + x^\circ) = \sin 30^\circ \sin 42^\circ \sin x^\circ$$

for x where $0 < x < 90$.

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13 The roots of the polynomial $x^4 - 4ix^3 + 3x^2 - 14ix - 44$ form the vertices of a parallelogram in the complex plane. What is the area of the parallelogram?

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14 Across the face of a rectangular post-it note, you idly draw lines that are parallel to its edges. Each time you draw a line, there is a 50% chance it'll be in each direction and you never draw over an existing line or the edge of the post-it note. After a few minutes, you notice that you've drawn 20 lines. What is the expected number of rectangles that the post-it note will be partitioned into?

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15 What is the smallest positive integer m such that $15!\, m$ can be expressed in more than one way as a product of 16 distinct positive integers, up to order?

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16 A snail begins a journey starting at the origin of a coordinate plane. The snail moves along line segments of length $\sqrt{10}$ and in any direction such that the horizontal and vertical displacements are both integers. As the snail moves, it leaves a trail tracing out its entire journey. After a while, this trail can form various polygons. What is the smallest possible area of a polygon that could be created by the snail's trail?

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17 Let O be the set of odd numbers between 0 and 100. Let T be the set of subsets of O of size 25. For any finite subset of integers S, let $P(S)$ be the product of the elements of S. Define $n = \sum_{S \in T} P(S)$. If you divide n by 17, what is the remainder?

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18 Let A be the locus of points (α, β, γ) in the $\alpha\beta\gamma$-coordinate space that satisfy the following properties:

(I) We have α, β, $\gamma > 0$.
(II) We have $\alpha + \beta + \gamma = \pi$.
(III) The intersection of the three cylinders in the xyz-coordinate space given by the equations

$$\begin{aligned} y^2 + z^2 &= \sin^2 \alpha \\ z^2 + x^2 &= \sin^2 \beta \\ x^2 + y^2 &= \sin^2 \gamma \end{aligned}$$

is nonempty.

Determine the area of A.

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19 Let S_- be the semicircular arc defined by

$$(x+1)^2 + (y - \frac{3}{2})^2 = \frac{1}{4} \text{ and } x \le -1.$$

Let S_+ be the semicircular arc defined by

$$(x-1)^2 + (y - \frac{3}{2})^2 = \frac{1}{4} \text{ and } x \ge 1.$$

Let R be the locus of points P such that P is the intersection of two lines, one of the form $Ax + By = 1$ where $(A, B) \in S_-$ and the other of the form $Cx + Dy = 1$ where $(C, D) \in S_+$. What is the area of R?

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20 Let $a_n = n(2n+1)$. Evaluate

$$\left| \sum_{1 \le j < k \le 36} \sin\left(\frac{\pi}{6}(a_k - a_j)\right) \right|.$$
