aka American High School Internet Mathematics Competition

www.artofproblemsolving.com/community/c3176431

by parmenides51

– Short Answer (each 1 point, total 40)

1 During the 2005 iTest, you will be introduced to Joe and Kathryn, two high school seniors. If J is the number of distinct permutations of JOE, and K is the number of distinct permutations of $KATHRYN$, find $K - J$.

2 Find the sum of the solutions of $x^3 + x + 182 = 0$.

3 Carrie, Miranda, Charlotte, and Samantha are sitting at a table with 5 numbered chairs (numbered 1 through 5). One chair is left open for Big, should he decide to join the four for lunch. In how many distinct ways can the four women occupy the table?

4 How many multiples of 2005 are factors of $(2005)^2$?

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$$\sin 30^o + \sin 45^o + \sin 60^o + \sin 90^o + \cos 120^o + \cos 135^o + \cos 150^o + \cos 180^o = ?$$

6 Kathryn, for a history project on sports, chronicled the history of college football. When she mentioned that Auburn got cheated out of the NCAA Football championship in the $2004 - 05$ season due to the many flaws in the BCS system, her teacher just couldn't contain her applause, and awarded an automatic A to her for the rest of the year.

The lecture was so popular, in fact, that many students pressed Kathryn to record the lecture on video and sell DVDs of it.

If the function for Kathryn's profit for selling DVDs of her college football presentation is $y = -x^2 + 14x + 251$, where y is Kathryn's profit and x is the price per DVD, what price (in dollars) will maximize her profit?

7 Find the coefficient of the fourth term of the expansion of $(x + y)^{15}$.

8 Joe and Kathryn work part-time jobs at the local mall to make some money for college. Joe works at GameStop, while Kathryn works at Bath and Body Works. However, neither of them usually ever leaves on pay day without spending a healthy portion of their check at their own store, especially angering Joe's parents, who think video games are for Neanderthals or children under 8.

Joe makes $\$8$ an hour, while Kathryn makes $\$10$ an hour. Both work 20 hours a week. Every week, Joe has a 20% probability of purchasing a used $\$25$ video game, and Kathryn has a 25% probability of purchasing a $\$30$ skin moisturizer. Find the expected value, in dollars, of their combined weekly ”take-home pay.” (Take-home pay is total pay minus in-store spending.)

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9 Find the area of the triangle with vertices of $(1,2)$, $(1,10)$, and $(5,5)$.

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10 The probability of U2 dismantling an atomic bomb is 11%. The probability of Coldplay finding X & Y is 23%. If the probability of both events occurring is $6\%,$ find the probability that neither occurs.

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11 Find the radius of the inscribed circle of a triangle with sides of length 40, 42, and 58.

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12 A sphere sits inside a cubic box, tangent on all 6 sides of the box. If a side of the box is 5, and the volume of the sphere is $x\pi$, find x.

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13 In a moment of impaired thought, Joe decides he wants to dress up as a member of NSYNC for his school Halloween party that night. If he dresses up as JC Chasez, he has a probability of 25% of getting beat up at the party. If he dresses up as Justin Timberlake, he has a 60% probability of getting beat up at the party. If he dresses up as any other member of NSYNC, he won’t get beat up because no one will recognize his costume. If there is an equal probability of him dressing up as any of the 5 NSYNC members, what is the probability he will get beat up at the Halloween party?

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14 A bottle contains 5 gallons of a 10% solution of oil. How many gallons of pure oil must be added to make a 30% oil solution? (round to the nearest hundredth)

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15 Kathryn has a crush on Joe. Dressed as Catwoman, she attends the same school Halloween party as Joe, hoping he will be there. If Joe gets beat up, Kathryn will be able to help Joe, and will be able to tell him how much she likes him. Otherwise, Kathryn will need to get her hipster friend, Max, who is DJing the event, to play Joe’s favorite song, ”Pieces of Me” by Ashlee Simpson, to get him out on the dance floor, where she’ll also be able to tell him how much she likes him. Since playing the song would be in flagrant violation of Max’s musical integrity as a DJ, Kathryn will have to bribe him to play the song. For every $\$10$ she gives Max, the probability of him playing the song goes up 10% (from 0% to 10% for the first $\$10$, from 10% to 20% for the next $\$10$, all the way up to 100% if she gives him $\$100$). Max only accepts money in increments of $\$10$. How much money should Kathryn give to Max to give herself at least a 65% chance of securing enough time to tell Joe how much she likes him?

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16 How many distinct integral solutions of the form (x,y) exist to the equation$21x+22y=43$ such that $1<x<11$ and $y<22$?

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17 On the 2004 iTest, we defined an *optimus* prime to be any prime number whose digits sum to a prime number. (For example, 83 is an optimus prime, because it is a prime number and its digits sum to 11, which is also a prime number.) Given that you select a prime number under 100, find the probability that is it not an optimus prime.

18 If the four sides of a quadrilateral are $2, 3, 6$, and x, find the sum of all possible integral values for x.

19 Find the amplitude of $y = 4\sin(x) + 3\cos(x)$.

20 If A is the 3×3 square matrix $\begin{bmatrix} 5 & 3 & 8 \\ 2 & 2 & 5 \\ 3 & 5 & 1 \end{bmatrix}$ and B is the 4×4 square matrix $\begin{bmatrix} 32 & 2 & 4 & 3 \\ 3 & 4 & 8 & 3 \\ 11 & 3 & 6 & 1 \\ 5 & 5 & 10 & 1 \end{bmatrix}$ find the sum of the determinants of A and B.

21 Two circles have a common internal tangent of length 17 and a common external tangent of length 25. Find the product of the radii of the two circles.

22 A regular n-gon has 135 diagonals. What is the measure of its exterior angle, in degrees? (An exterior angle is the supplement of an interior angle.)

23 $\sqrt[3]{x + \sqrt[3]{x + \sqrt[3]{x + \sqrt[3]{x...}}}} = 8$. Find x.

24 SQUARING OFF: Master Chief and Samus Aran take turns firing rockets at one another from across the Cartesian plane. Master Chief's movement is restricted to lattice points within the 10×10 square with vertices $(0, 0)$, $(0, 10)$, $(10, 0)$, and $(10, 10)$, while Samus Aran's movement is restricted to lattice points inside the 10×10 square with vertices $(0, 0)$, $(-10, 0)$, $(0, -10)$, and $(-10, -10)$. Neither player can be located on or beyond the border of his or her square. Both players randomly choose a lattice point at which they begin the game, and do not move the rest of the game (until either they are killed or kill the other player).

Each player's turn consists of firing a rocket, targeted at a specific undestroyed lattice point inside the border of the opponent's movement square, which hits immediately. When a rocket hits its intended lattice point, it explodes, destroying the surrounding 3×3 square (8 additional adjacent lattice points).

The game ends when one player is hit by a rocket (when the player is located within the 3×3 grid hit by a rocket). If the highest possible probability that Samus Aran wins the game in three turns or less, assuming Master Chief goes first, is expressed as a/b, where a and b are relatively prime integers, find $a + b$.

25 Consider the set $\{1!, 2!, 3!, 4!, \ldots, 2004!, 2005!\}$. How many elements of this set are divisible by 2005?

26 Joe and Kathryn are both on the school math team, which practices every Wednesday after school until 4 PM for competitions. The team was preparing for the 2005 iTest when Joe realized how crazy he was for not asking Kathryn out – the way she worked those iTest problems, solving question after question, almost made him go insane sitting there that day. He never felt the same way when she worked on preparing for other competitions – they just aren't the same.

Kathryn always beat Joe at competitions, too. Joe admired her resolve and unwillingness to make herself look stupid, when so many other girls he knew at school tried to pretend they were stupid in order to attract guys.

So as time ticked away and that afternoon's Wednesday practice neared an end, Joe was determined to strike up a conversation with Kathryn and ask her out. He really wanted to impress her, so he thought he'd ask her a really hard history of math question that she didn't know. Naturally, she'd want the answer, and be so impressed with Joe's brilliance that she'd go out with him on Friday night.

Great plan. Seriously.

When Joe asked Kathryn after class, "Who was the mathematician that died in approximately 200 B.C. that developed a method for calculating all prime numbers?" Kathryn gave the correct response. What name did she say?

27 Find the sum of all non-zero digits that can repeat at the end of a perfect square. (For example, if 811 were a perfect square, 1 would be one of these non-zero digits.)

28 Yoknapatawpha County has $500,000$ families. Each family is expected to continue to have children until it has a girl, at which point each family stops having children. If the probability of having a boy is 50%, and no families have either fertility problems or multiple children per birthing, how many families are expected to have at least 5 children?

29 WHY is a triangle with angle $W \geq 90$ degrees. On the side HY, two distinct points M and E are chosen such that angle HWM is equivalent to angle MWE and $HM * YE = HY * ME$. Find the angle MWY.

30 How many of the following statements are false?
a. 2005 distinct positive integers exist such that the sum of their squares is a cube and the sum of their cubes is a square.
b. There are 2 integral solutions to $x^2 + y^2 + z^2 = x^2y^2$.
c. If the vertices of a triangle are lattice points in a plane, the diameter of the triangle's circumcircle will never exceed the product of the triangle's side lengths.

31 Let $X = 123456789$. Find the sum of the tens digits of all integral multiples of 11 that can be obtained by interchanging two digits of X.

32 Find the shortest distance between the points $(3,5)$ and $(7,8)$.

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33 If the coefficient of the third term in the binomial expansion of $(1-3x)^{1/4}$ is $-a/b$, where a and b are relatively prime integers, find $a+b$.

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34 If x is the number of solutions to the equation $a^2+b^2+c^2=d^2$ of the form (a,b,c,d) such that $\{a,b,c\}$ are three consecutive square numbers and d is also a square number, find x.

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35 How many values of x satisfy the equation

$$(x^2-9x+19)^{x^2+16x+60}=1?$$

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36 Find the determinant of this matrix: $\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 & 2 \\ 4 & 4 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 2 & 2 \\ 4 & 4 & 4 & 4 & 4 & 2 \end{bmatrix}$

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37 How many zeroes appear at the end of 209 factorial?

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38 LeBron James and Carmelo Anthony play a game of one-on-one basketball where the first player to 3 points or more wins. LeBron James has a 20% chance of making a 3-point shot; Carmelo has a 10% chance of making a 3-pointer. LeBron has a 40% chance of making a 2-point shot from anywhere inside the 3-point line (excluding dunks, which are also worth 2 points); Carmelo has a 52% chance of making a 2-point shot from anywhere inside the 3-point line (excluding dunks). LeBron has a 90% chance of dunking on Carmelo; Carmelo has a 95% chance of dunking on LeBron. If each player has 3 possessions to try to win, LeBron James goes first, and both players follow a rational strategy to try to win, what is the probability that Carmelo Anthony wins the game?

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39 What is the smallest positive integer that when raised to the 6^{th} power, it can be represented by a sum of the 6^{th} powers of distinct smaller positive integers?

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40 $ITEST + AHSIMC = 6666CS$. Each letter represents a unique digit from 0 to 9. How many solutions of the form (C,A,S,H) exist?

– Chain Reaction (20 points total)

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1 **1A.** The iTest, by virtue of being the first national internet-based high school math competition, saves a lot of paper every year. The quantity of trees saved ("a") is determined by the following formula: $a = x^2 + 3x + 9$, where x is the number of participating students in the competition. If x is the correct answer from short answer x=20, then find a. *(1 point)*

1B. Let q be the sum of the digits of a. If $q = b! - (b-1)! + (b-2)! - (b-3)!$, find b. *(2 points)*

1C. Find the number of the following statements that are false: *(4 points)*

1. q is the first prime number resulting from the sum of cubes of distinct fractions, where both the numerator and denominator are primes.
2. q is composite.
3. q is composite and is the sum of the first four prime numbers and 1.
4. q is the smallest prime equal to the difference of cubes of two consecutive primes.
5. q is not the smallest prime equal to the product of twin primes plus their arithmetic mean.
6. The sum of q consecutive Fibonacci numbers, starting from the q^{th} Fibonacci number, is prime.
7. q is the largest prime factor of $1bbb$.
8. q is the 8^{th} largest prime number.
9. a is composite.
10. $a + q + b = q^2$.
11. The decimal expansion of q^q begins with q.
12. q is the smallest prime equal to the sum of three distinct primes.
13. $q^5 + q^2 + q^1 + q^3 + q^5 + q^6 + q^4 + q^0 = 52135640.$
14. q is not the smallest prime such that q and q^2 have the same sum of their digits.
15. q is the smallest prime such that q = (the product of its digits + the sum of its digits).

1A. 469
1B. 4
1C. 6

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2 **2A.** Two triangles ABC and ABD share a common side. ABC is drawn such that its entire area lies inside the larger triangle ABD. If $AB = 20$, side AD meets side AB at a right angle, and point C is between points A and D, then find the area outside of triangle ABC but within ABD, given that both triangles have integral side lengths and AB is the smallest side of either triangle. ABC and ABD are both primitive right triangles. *(1 point)*

2B. Find the sum of all positive integral factors of the correct answer to **2A**. *(2 points)*

2C. Let B be the sum of the digits of the correct answer to **2B** above. If the solution to the functional equation $21 * f(x) - 7 * f(1/x) = Bx$ is of the form $(Ax^2 + C)/Dx$, find C, given that A, C, and D are relatively prime (they don't share a common prime factor). *(3 points)*

2A.780
2B. 2352
2C. 3

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3 **3A.** Sudoku, the popular math game that caught on internationally before making its way here to the United States, is a game of logic based on a grid of 9 rows and 9 columns. This grid is subdivided into 9 squares ("subgrids") of length 3. A successfully completed Sudoku puzzle fills this grid with the numbers 1 through 9 such that each number appears only once in each row, column, and individual 3×3 subgrid. Each Sudoku puzzle has one and only one correct solution.

Complete the following Sudoku puzzle, and find the sum of the numbers represented by X, Y, and Z in the grid. *(1 point)*

		2	9	7	4			
	Z						5	7
						Y		
		4		5				2
		9	X	1		6		
8				3		4		
1	3							
			6	8	2	9		

3B. Let A equal the correct answer from **3A**. In triangle WXY, $tan\angle YWX = (A+8)/.5A$, and the altitude from W divides XY into segments of 3 and $A+3$. What is the sum of the digits of the square of the area of the triangle? *(2 points)*

3C. Let B equal the correct answer from **3B**. If a student team taking the 2005 iTest solves B problems correctly, and the probability that this student team makes over a 18 is x/y where x and y are relatively prime, find $x+y$.

Assume that each chain reaction question – all 3 parts it contains – counts as a single problem. Also assume that the student team does not attempt any tiebreakers. *(4 points)*

[i][Note for problem 3C beacuse you might not know how points are given at that iTest:
Part A (aka Short Answer), has 40 problems of 1 point each, total 40
Part B (aka Chain Reaction), has 3 problems of 7,6,7 points each, total 20
Part C (aka Long Answer), has 5 problems of 8 point each, total 40
all 3 parts add to 100 points totally (here (`https://artofproblemsolving.com/community/c3176431_itest_2005`) is that test)][/i]

3A.14
3B. 4
3C. 6563

– Long Answer (each 8 point, total 40)

1 Joe finally asked Kathryn out. They go out on a date on a Friday night, racing at the local go-kart track. They take turns racing across an 8×8 square grid composed of 64 unit squares. If Joe and Kathryn start in the lower left-hand corner of the 8×8 square, and can move either up or right along any side of any unit square, what is the probability that Joe and Kathryn take the same exact path to reach the upper right-hand corner of the 8×8 square grid?

2 $f(0) = 0$ $f(1) = 1$ $f(2) = 3$ $f(3) = 5$ $f(4) = 9$ $f(5) = 11$ $f(6) = 29$ $f(11) = 31$ $f(20) = ?$

3 For a convex hexagon $AHSIMC$ whose side lengths are all 1, let Z and z be the maximum and minimum values, respectively, of the three diagonals AI, HM, and SC. If $\sqrt{x} \le Z \le \sqrt{y}$ and $\sqrt{q} \le z \le \sqrt{r}$, find the product $qrxy$, if q,r, x, and y are all integers.

4 The function f is defined on the set of integers and satisfies • $f(n) = n - 2$, if $n \ge 2005$ • $f(n) = f(f(n+7))$, if $n < 2005$.
Find $f(3)$.

5 The following is a code and is meant to be broken.

2 707 156 377 38 2 328 17 185 2 713 73 566 1130 328 73 38 259 471 38 17 566 2 134 707 38 274 377 328 38 1130 40 377 566 73 820 566 566 134 11 2 328 38 185 2 713 566 134 328 2 918 134 11 713 134 274 707 713 73 38 1130 17 134 707 11 820 707 707 38 17 713 73 38 134 566 40 2 918 377 566 134 713 38 328 820 274 4 38 566 707

156 377 38 707 40 2 918 377 566 134 713 38 328 820 274 4 38 566 134 707 713 73 38 2 328 707 991 38 566 713 377 713 73 38 707 38 918 38 328 713 73 707 73 377 566 713 2 328 707 991 38 566 532 820 38 707 713 134 377 328 377 328 713 73 134 707 713 38 707 713

185 2 713 73 566 1130 328 707 40 2 918 377 566 134 713 38 328 820 274 4 38 566 134 707 713 73 38 2 328 707 991 38 566 713 377 713 73 38 707 38 11 377 328 17 259 377 328 79 2 328 707 991 38 566 532 820 38 707 713 134 377 328 377 328 713 73 134 707 713 38 707 713

991 73 2 713 134 707 713 73 38 707 820 274 377 40 713 73 38 134 566 40 2 918 377 566 134 713 38 328 820 274 4 38 566 707

– Tiebreakers (points as indicated)

1 Find the number of distinct permutations of $ITEST$.

(.1 point)

2 When $1^0 + 2^1 + 3^2 + ... + 100^{99}$ is divided by 5, a remainder of N is obtained such that N is between 0 and 4 inclusive. Find N.

(.1 point)

3 Find the probability that any given row in Pascal's Triangle contains a perfect square.

(.1 point)

4 If the product of $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})(-\sqrt{2}+\sqrt{3}+\sqrt{5})$ is $12\sqrt{6}+6\sqrt{x}$, find x.

(*0 points* - **THROWN OUT**)

5 Find the sum of the answers to all even numbered Short Answer problems, with the exception of#26, rounded to the nearest tenth.

(.7 points)