2022 United States Ersatz Math Olympiad, 22-23 October 2022

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Day 1 October 22, 2022

1 A *stick* is defined as a $1 \times k$ or $k \times 1$ rectangle for any integer $k \geq 1$. We wish to partition the cells of a 2022×2022 chessboard into m non-overlapping sticks, such that any two of these m sticks share at most one unit of perimeter. Determine the smallest m for which this is possible.

Holden Mui

2 A function $\psi \colon \mathbb{Z} \to \mathbb{Z}$ is said to be *zero-requiem* if for any positive integer n and any integers a_1, ..., a_n (not necessarily distinct), the sums $a_1 + a_2 + \dots + a_n$ and $\psi(a_1) + \psi(a_2) + \dots + \psi(a_n)$ are not both zero.

Let f and g be two zero-requiem functions for which $f \circ g$ and $g \circ f$ are both the identity function (that is, f and g are mutually inverse bijections). Given that $f + g$ is *not* a zero-requiem function, prove that $f \circ f$ and $g \circ g$ are both zero-requiem.

Sutanay Bhattacharya

3 Point P lies in the interior of a triangle ABC. Lines AP, BP, and CP meet the opposite sides of triangle ABC at A', B', and C' respectively. Let P_A the midpoint of the segment joining the incenters of triangles BPC' and CPB', and define points P_B and P_C analogously. Show that if

$$AB' + BC' + CA' = AC' + BA' + CB'$$

then points $P, P_A, P_B,$ and P_C are concyclic.

Nikolai Beluhov

Day 2 October 23, 2022

4 Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points P, Q, R, S lie in the interiors of segments $AB, BC, CD, DA,$ respectively, such that

$$\angle PDA = \angle PCB, \ \angle QAB = \angle QDC, \ \angle RBC = \angle RAD, \text{ and } \angle SCD = \angle SBA.$$

Let AQ intersect BS at X, and DQ intersect CS at Y. Prove that lines PR and XY are either parallel or coincide.

Tilek Askerbekov

5 Let $\tau(n)$ denote the number of positive integer divisors of a positive integer n (for example, $\tau(2022) = 8$). Given a polynomial $P(X)$ with integer coefficients, we define a sequence $a_1, a_2, \dots$ of nonnegative integers by setting

$$a_n = \begin{cases} \gcd(P(n), \tau(P(n))) & \text{if } P(n) > 0 \\ 0 & \text{if } P(n) \le 0 \end{cases}$$

for each positive integer n. We then say the sequence *has limit infinity* if every integer occurs in this sequence only finitely many times (possibly not at all).

Does there exist a choice of $P(X)$ for which the sequence a_1, a_2, . . . has limit infinity?

Jovan Vuković

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6 Find all positive integers k for which there exists a nonlinear function $f : \mathbb{Z} \to \mathbb{Z}$ such that the equation

$$f(a) + f(b) + f(c) = \frac{f(a-b) + f(b-c) + f(c-a)}{k}$$

holds for any integers a, b, c satisfying $a + b + c = 0$ (not necessarily distinct).

Evan Chen
