Korea National Olympiad 2022

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1 Three sequences a_n, b_n, c_n satisfy the following conditions.

-$a_1 = 2,\ b_1 = 4,\ c_1 = 5$
-$\forall n,\ a_{n+1} = b_n + \frac{1}{c_n},\ b_{n+1} = c_n + \frac{1}{a_n},\ c_{n+1} = a_n + \frac{1}{b_n}$

Prove that for all positive integers n, $\ max(a_n, b_n, c_n) > \sqrt{2n+13}$.

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2 In a scalene triangle ABC, let the angle bisector of A meets side BC at D. Let E, F be the circumcenter of the triangles ABD and ADC, respectively. Suppose that the circumcircles of the triangles BDE and DCF intersect at $P(\neq D)$, and denote by O, X, Y the circumcenters of the triangles ABC, BDE, DCF, respectively. Prove that OP and XY are parallel.

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3 Suppose that the sequence $\{a_n\}$ of positive integers satisfies the following conditions:

-For an integer $i \geq 2022$, define a_i as the smallest positive integer x such that $x + \sum_{k=i-2021}^{i-1} a_k$ is a perfect square.
-There exists infinitely many positive integers n such that $a_n = 4 \times 2022 - 3$.

Prove that there exists a positive integer N such that $\sum_{k=n}^{n+2021} a_k$ is constant for every integer $n \geq N$.
And determine the value of $\sum_{k=N}^{N+2021} a_k$.

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4 For positive integers m, n $(m > n)$, $a_{n+1}, a_{n+2}, ..., a_m$ are non-negative integers that satisfy the following inequality.

$$2 > \frac{a_{n+1}}{n+1} \geq \frac{a_{n+2}}{n+2} \geq \cdots \geq \frac{a_m}{m}$$

Find the number of pair $(a_{n+1}, a_{n+2}, \cdots, a_m)$.

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5 For a scalene triangle ABC with an incenter I, let its incircle meets the sides BC, CA, AB at D, E, F, respectively. Denote by P the intersection of the lines AI and DF, and Q the intersection of the lines BI and EF. Prove that $\overline{PQ} = \overline{CD}$.

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6 $n(\geq 4)$ islands are connected by bridges to satisfy the following conditions:

-Each bridge connects only two islands and does not go through other islands.
-There is at most one bridge connecting any two different islands.
-There does not exist a list $A_1, A_2, \ldots, A_{2k} (k \geq 2)$ of distinct islands that satisfy the following:

For every $i = 1, 2, \dots, 2k$, the two islands A_i and A_{i+1} are connected by a bridge. (Let $A_{2k+1} = A_1$)

Prove that the number of the bridges is at most $\frac{3(n-1)}{2}$.

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7 Suppose that the sequence $\{a_n\}$ of positive reals satisfies the following conditions:

-$a_i \leq a_j$ for every positive integers $i < j$.
-For any positive integer $k \geq 3$, the following inequality holds:

$$(a_1 + a_2)(a_2 + a_3) \cdots (a_{k-1} + a_k)(a_k + a_1) \leq (2^k + 2022) a_1 a_2 \cdots a_k$$

Prove that $\{a_n\}$ is constant.

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8 p is a prime number such that its remainder divided by 8 is 3. Find all pairs of rational numbers (x, y) that satisfy the following equation.

$$p^2 x^4 - 6px^2 + 1 = y^2$$
