Saudi Arabia IMO Team Selection Test 2022

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by parmenides51, amuthup, v_Enhance, CyclicISLscelesTrapezoid, admechii

– Day I

1 Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

2 Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D meet rays BA and BC at E and F, respectively. A point T is chosen inside $\triangle ABC$ so that $\overline{TE} \parallel \overline{CD}$ and $\overline{TF} \parallel \overline{AD}$. Let $K \neq D$ be a point on segment DF satisfying $TD = TK$. Prove that lines AC, DT, and BK are concurrent.

3 Find all non-constant functions $f : Q^+ \to Q^+$ satisfying the equation

$$f(ab + bc + ca) = f(a)f(b) + f(b)f(c) + f(c)f(a)$$

for all $a, b, c \in Q^+$.

– Day II

1 Let (a_n) be the integer sequence which is defined by $a_1 = 1$ and

$$a_{n+1} = a_n^2 + n \cdot a_n \ , \ \forall n \geq 1.$$

Let S be the set of all primes p such that there exists an index i such that $p|a_i$.
Prove that the set S is an infinite set and it is not equal to the set of all primes.

2 For each integer $n \geq 1,$ compute the smallest possible value of

$$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$

over all permutations $(a_1, \dots, a_n)$ of $\{1, \dots, n\}$.

Proposed by Shahjalal Shohag, Bangladesh

3 The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a [i]path from X to Y[/i] is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

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Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Proposed by Warut Suksompong, Thailand

– Day IIII

1 Which positive integers n make the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\lfloor \frac{ij}{n+1} \right\rfloor = \frac{n^2(n-1)}{4}$$

true?

2 Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ *neighbors* - namely, the k students closest to S on the left, and the k students closest to S on the right.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbors.

Proposed by Gurgen Asatryan, Armenia

3 Show that $n! = a^{n-1} + b^{n-1} + c^{n-1}$ has only finitely many solutions in positive integers.

Proposed by Dorlir Ahmeti, Albania

– Day IV

1 There are a) 2022, b) 2023 plates placed around a round table and on each of them there is one coin. Alice and Bob are playing a game that proceeds in rounds indefinitely as follows. In each round, Alice first chooses a plate on which there is at least one coin. Then Bob moves one coin from this plate to one of the two adjacent plates, chosen by him. Determine whether it is possible for Bob to select his moves so that, no matter how Alice selects her moves, there are never more than two coins on any plate.

2 Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for $i = 1, 2, \ldots, k$, the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

3 Let A, B, C, D be points on the line d in that order and $AB = CD$. Denote (P) as some circle that passes through A, B with its tangent lines at A, B are a, b. Denote (Q) as some circle that passes through C, D with its tangent lines at C, D are c, d. Suppose that a cuts c, d at K, L respectively and b cuts c, d at M, N respectively. Prove that four points K, L, M, N belong to a same circle (ω) and the common external tangent lines of circles (P), (Q) meet on (ω).
