Greece Team Selection Test 2022

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1 Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that $a^2 + b + 3$ is not divisible by the cube of any prime, and

$$n = \frac{ab + 3b + 8}{a^2 + b + 3}.$$

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2 Consider triangle ABC with $AB < AC < BC$, inscribed in triangle Γ_1 and the circles $\Gamma_2(B, AC)$ and $\Gamma_2(C, AB)$. A common point of circle Γ_2 and Γ_3 is point E, a common point of circle Γ_1 and Γ_3 is point F and a common point of circle Γ_1 and Γ_2 is point G, where the points E, F, G lie on the same semiplane defined by line BC, that point A doesn't lie in. Prove that circumcenter of triangle EFG lies on circle Γ_1.

Note: By notation $\Gamma(K, R)$, we mean random circle Γ has center K and radius R.

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3 Find largest possible constant M such that, for any sequence a_n, $n = 0, 1, 2, ...$ of real numbers, that satisfies the conditions :
i) $a_0 = 1$, $a_1 = 3$
ii) $a_0 + a_1 + ... + a_{n-1} \geq 3a_n - a_{n+1}$ for any integer $n \geq 1$
to be true that

$$\frac{a_{n+1}}{a_n} > M$$

for any integer $n \geq 0$.

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4 In an exotic country, the National Bank issues coins that can take any value in the interval $[0, 1]$. Find the smallest constant $c > 0$ such that the following holds, no matter the situation in that country:

[i]Any citizen of the exotic country that has a finite number of coins, with a total value of no more than 1000, can split those coins into 100 boxes, such that the total value inside each box is at most c.[/i]
