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by parmenides51, ericxyzhu

—   Day 1

1   How many triples (a, b, c) with $a, b, c \in \mathbb{R}$ satisfy the following system?

$$\begin{cases} a^4 - b^4 = c \\ b^4 - c^4 = a \\ c^4 - a^4 = b \end{cases}$$

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2   Anselmo and Claudio are playing alternatively a game with fruits in a box. The box initially has 32 fruits. Anselmo plays first and each turn consists of taking away $1, 2$ or 3 fruits from the box or taking away $\frac{2}{3}$ of the fruits from the box (this is only possible when the number of the fruits left in the box is a multiple of 3). The player that takes away the last fruit from the box wins. Which of these two players has a winning strategy? How should that player play in order to win?

3   The positive integers x and y are such that $x^{2022} + x + y^2$ is divisible by xy.

a) Give an example of such integers x and y, with $x > y$.

b) Prove that x is a perfect square.

—   Day 2

4   How many integer solutions exist that satisfy this equation?

$$x + 4y - 343\sqrt{x} - 686\sqrt{y} + 4\sqrt{xy} + 2022 = 0$$

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5   Tow circumferences of radius R_1 and R_2 are tangent externally between each other. Besides that, they are both tangent to a semicircle with radius of 1, as shown in the figure. (Diagram is in the attachment)

a) If A_1 and A_2 are the tangency points of the two circumferences with the diameter of the semicircle, find the length of $\overline{A_1 A_2}$.

b) Prove that $R_1 + R_2 = 2\sqrt{R_1 R_2}(\sqrt{2} - \sqrt{R_1 R_2})$.

6   A necklace contains 2024 pearls, each one of them having one of the following colours: black, green and yellow. Each moment, we will switch each one of all pearls simultaneously to a new one following the following rules:

i) If its two neighbours are of the same colour, then it'll be switched to that same colour.
ii) If its two neighbours are of different colours, then it'll be switched to the third colour.

a) Does there exist any necklace that can be transformed into a necklace that consists of only yellow pearls if initially half of the pearls are black and the other half is green?

b) Does there exist a necklace that can be transformed into a necklace that consists of only yellow pearls if initially 998 pearls are black and the rest 1026 pearls are green?