KJMO 2022

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by parmenides51, pseudo1, Olympiadium

– day 1

1 The inscribed circle of an acute triangle ABC meets the segments AB and BC at D and E respectively. Let I be the incenter of the triangle ABC. Prove that the intersection of the line AI and DE is on the circle whose diameter is AC(passing through A, C).

3 For a given odd prime number p, define $f(n)$ the remainder of d divided by p, where d is the biggest divisor of n which is not a multiple of p. For example when $p=5$, $f(6)=1, f(35)=2, f(75)=3$. Define the sequence $a_1, a_2, \ldots, a_n, \ldots$ of integers as the followings:

-$a_1=1$
-$a_{n+1}=a_n+(-1)^{f(n)+1}$ for all positive integers n.

Determine all integers m, such that there exist infinitely many positive integers k such that $m=a_k$.

4 Find all function $f:\mathbb{N}\longrightarrow\mathbb{N}$ such that

forall positive integers x and y, $\frac{f(x+y)-f(x)}{f(y)}$ is again a positive integer not exceeding 2022^{2022}.

– day 2

5 A sequence of real numbers $a_1, a_2, \ldots$ satisfies the following conditions. $a_1=2$, $a_2=11$.
for all positive integer n, $2a_{n+2}=3a_n+\sqrt{5(a_n^2+a_{n+1}^2)}$
Prove that a_n is a rational number for each of positive integer n.

6 Let ABC be a isosceles triangle with $\overline{AB}=\overline{AC}$. Let $D(\neq A, C)$ be a point on the side AC, and circle Ω is tangent to BD at point E, and AC at point C. Denote by $F(\neq E)$ the intersection of the line AE and the circle Ω, and $G(\neq a)$ the intersection of the line AC and the circumcircle of the triangle ABF. Prove that points D, E, F, and G are concyclic.

7 Consider n cards with marked numbers 1 through n. No number have repeted, namely, each number has marked exactly at one card. They are distributed on n boxes so that each box contains exactly one card initially. We want to move all the cards into one box all together according to the following instructions
The instruction: Choose an integer $k(1\leq k\leq n)$, and move a card with number k to the other box such that sum of the number of the card in that box is multiple of k.
Find all positive integer n so that there exists a way to gather all the cards in one box.

Thanks to @scnwust for correcting wrong translation.

8 Find all pairs (x, y) of rational numbers such that

$$xy^2 = x^2 + 2x - 3$$