Mexico National Olympiad 2022

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1 A number x is "Tlahuica" if there exist prime numbers $p_1,\ p_2,\ \dots,\ p_k$ such that
$$x = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k}.$$
Find the largest Tlahuica number x such that $0 < x < 1$ and there exists a positive integer $m \leq 2022$ such that mx is an integer.

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2 Let n be a positive integer. David has six $n \times n$ chessboards which he arranges in an $n \times n \times n$ cube. Two cells are "aligned" if they can be connected by a path of cells $a = c_1,\ c_2,\ \dots,\ c_m = b$ such that all consecutive cells in the path share a side, and the sides that the cell c_i shares with its neighbors are on opposite sides of the square for $i = 2,\ 3,\ \dots\ m-1$.

Two towers attack each other if the cells they occupy are aligned. What is the maximum amount of towers he can place on the board such that no two towers attack each other?

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3 Let $n > 1$ be an integer and $d_1 < d_2 < \dots < d_m$ the list of its positive divisors, including 1 and n. The m instruments of a mathematical orchestra will play a musical piece for m seconds, where the instrument i will play a note of tone d_i during s_i seconds (not necessarily consecutive), where d_i and s_i are positive integers. This piece has "sonority" $S = s_1 + s_2 + \dots s_n$.

A pair of tones a and b are harmonic if $\frac{a}{b}$ or $\frac{b}{a}$ is an integer. If every instrument plays for at least one second and every pair of notes that sound at the same time are harmonic, show that the maximum sonority achievable is a composite number.

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4 Let n be a positive integer. In an $n \times n$ garden, a fountain is to be built with 1×1 platforms covering the entire garden. Ana places all the platforms at a different height. Afterwards, Beto places water sources in some of the platforms. The water in each platform can flow to other platforms sharing a side only if they have a lower height. Beto wins if he fills all platforms with water.
Find the least number of water sources that Beto needs to win no matter how Ana places the platforms.

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5 Let $n > 1$ be a positive integer and $d_1 < d_2 < \dots < d_m$ be its m positive divisors, including 1 and n. Lalo writes the following $2m$ numbers on a board:
$$d_1, d_2 \dots, d_m, d_1 + d_2, d_2 + d_3, \dots, d_{m-1} + d_m, N$$
where N is a positive integer. Afterwards, Lalo erases any number that is repeated (for example, if a number appears twice, he erases one of them). Finally, Lalo realizes that the numbers left on the board are exactly all the divisors of N. Find all possible values that n can take.

6 Find all integers $n \geq 3$ such that there exists a convex n-gon $A_1A_2\dots A_n$ which satisfies the following conditions:
- All interior angles of the polygon are equal
- Not all sides of the polygon are equal
- There exists a triangle T and a point O inside the polygon such that the n triangles $OA_1A_2,\ OA_2A_3,\ \dots,\ OA$
are all similar to T, not necessarily in the same vertex order.