AMC 12/AHSME 2022

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by peace09, programmeruser, MathArt4, mannshah1211, popcorn1, sugar_rush, Geometry285, serichaoo, donian9265, ihatemath123, fukano_2, JimY, DottedCaculator, cosmicgenius, fruitmonster97, AngelaLZ, NegativeZeroPlusOne, Charles3829, Spectator, JN5537, kc5170, Professor-Mom, naman12, brainfertilzer, inventivedant, rrusczyk

– A

– November 10, 2022

1 What is the value of

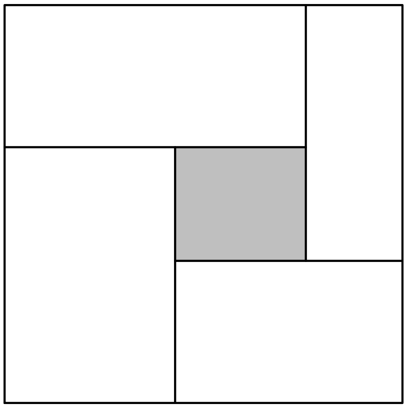
$$3+\frac{1}{3+\frac{1}{3+\frac{1}{3}}}?$$

(A) $\frac{31}{10}$ **(B)** $\frac{49}{15}$ **(C)** $\frac{33}{10}$ **(D)** $\frac{109}{33}$ **(E)** $\frac{15}{4}$

2 The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

3 Five rectangles, A, B, C, D, and E, are arranged in a square as shown below. These rectangles have dimensions 1×6, 2×4, 5×6, 2×7, and 2×3, respectively. (The figure is not drawn to scale.) Which of the five rectangles is the shaded one in the middle?



(A) A **(B)** B **(C)** C **(D)** D **(E)** E

4 The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

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5 Let the *taxicab distance* between points (x_1, y_1) and (x_2, y_2) in the coordinate plane is given by $|x_1 - x_2| + |y_1 - y_2|$. For how many points P with integer coordinates is the taxicab distance between P and the origin less than or equal to 20?

(A) 441 **(B)** 761 **(C)** 841 **(D)** 921 **(E)** 924

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6 A data set consists of 6 (not distinct) positive integers: $1, 7, 5, 2, 5$, and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X?

(A) 10 **(B)** 26 **(C)** 32 **(D)** 36 **(E)** 40

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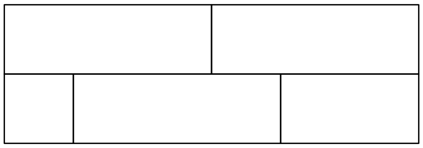
7 A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



(A) 120 **(B)** 270 **(C)** 360 **(D)** 540 **(E)** 720

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8 The infinite product

$$\sqrt[3]{10} \cdot \sqrt[3]{\sqrt[3]{10}} \cdot \sqrt[3]{\sqrt[3]{\sqrt[3]{10}}} \cdots$$

evaluates to a real number. What is that number?

(A) $\sqrt{10}$ **(B)** $\sqrt[3]{100}$ **(C)** $\sqrt[4]{1000}$ **(D)** 10 **(E)** $10\sqrt[3]{10}$

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9 On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

(A) 7 **(B)** 12 **(C)** 21 **(D)** 27 **(E)** 31

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10 What is the number of ways the numbers from 1 to 14 can be split into 7 pairs such that for each pair, the greater number is at least 2 times the smaller number?

(A) 108 **(B)** 120 **(C)** 126 **(D)** 132 **(E)** 144

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11 What is the product of all real numbers x such that the distance on the number line between $\log_6 x$ and $\log_6 9$ is twice the distance on the number line between $\log_6 10$ and 1?

(A) 10 **(B)** 18 **(C)** 25 **(D)** 36 **(E)** 81

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12 Let M be the midpoint of $\overline{AB}$ in regular tetrahedron $ABCD$. What is $\cos(\angle CMD)$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{\sqrt{3}}{2}$

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13 Let $\mathcal{R}$ be the region in the complex plane consisting of all complex numbers z that can be written as the sum of complex numbers z_1 and z_2, where z_1 lies on the segment with endpoints 3 and $4i$, and z_2 has magnitude at most 1. What integer is closest to the area of $\mathcal{R}$?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

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14 What is the value of

$$(\log 5)^3 + (\log 20)^3 + (\log 8)(\log 0.25)$$

where $\log$ denotes the base-ten logarithm?

(A) $\frac{3}{2}$ **(B)** $\frac{7}{4}$ **(C)** 2 **(D)** $\frac{9}{4}$ **(E)** 3

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15 The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism. A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

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16 A *triangular number* is a positive integer that can be expressed in the form $t_n = 1+2+3+\cdots+n$, for some positive integer n. The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2$, $t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 18 **(E)** 27

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17 Suppose a is a real number such that the equation

$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a can be written in the form $(p, q) \cup (q, r)$, where p, q, and r are real numbers with $p < q < r$. What is $p + q + r$?

(A) -4 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 4

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18 Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations transformations $T_1, T_2, T_3, \dots, T_n$ returns the point $(1, 0)$ back to itself?

(A) 359 **(B)** 360 **(C)** 719 **(D)** 720 **(E)** 721

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19 Suppose that 13 cards numbered $1, 2, 3, \dots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards 1, 2, 3 are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, 7, 8, 9, 10 on the fourth pass, and 11, 12, 13 on the fifth pass. For how many of the $13!$ possible orderings of the cards will the 13 cards be picked up in exactly two passes?

7	11	8	6	4	5	9	12	1	13	10	2	3

(A) 4082 **(B)** 4095 **(C)** 4096 **(D)** 8178 **(E)** 8191

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20 Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3,$ and $PD = 4$. What is $\frac{BC}{AD}$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

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21 Let $P(x) = x^{2022} + x^{1011} + 1$. Which of the following polynomials divides $P(x)$?

(A) $x^2 - x + 1$ **(B)** $x^2 + x + 1$ **(C)** $x^4 + 1$ **(D)** $x^6 - x^3 + 1$ **(E)** $x^6 + x^3 + 1$

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22 Let c be a real number, and let z_1, z_2 be the two complex numbers satisfying the quadratic $z^2 - cz + 10 = 0$. Points $z_1, z_2, \frac{1}{z_1}$, and $\frac{1}{z_2}$ are the vertices of a (convex) quadrilateral Q in the complex plane. When the area of Q obtains its maximum value, c is the closest to which of the following?

(A) 4.5 **(B)** 5 **(C)** 5.5 **(D)** 6 **(E)** 6.5

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23 Let h_n and k_n be the unique relatively prime positive integers such that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\frac{h_n}{k_n}.$$

Let L_n denote the least common multiple of the numbers $1, 2, 3, \cdots, n$. For how many integers n with $1 \le n \le 22$ is $k_n < L_n$?

(A) 0 **(B)** 3 **(C)** 7 **(D)** 8 **(E)** 10

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24 How many strings of length 5 formed from the digits 0,1,2,3,4 are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j? (For example, 02214 satisfies the condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

(A) 500 **(B)** 625 **(C)** 1089 **(D)** 1199 **(E)** 1296

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25 A circle with integer radius r is centered at (r, r). Distinct line segments of length c_i connect points $(0, a_i)$ to $(b_i, 0)$ for $1 \le i \le 14$ and are tangent to the circle, where a_i, b_i, and c_i are all positive integers and $c_1 \le c_2 \le \cdots \le c_{14}$. What is the ratio $\frac{c_{14}}{c_1}$ for the least possible value of r?

(A) $\frac{21}{5}$ **(B)** $\frac{85}{13}$ **(C)** 7 **(D)** $\frac{39}{5}$ **(E)** 17

– B

– November 16, 2022

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1 Define $x \diamond y$ to be $|x - y|$ for all real numbers x and y. What is the value of

$$(1 \diamond (2 \diamond 3)) - ((1 \diamond 2) \diamond 3)?$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

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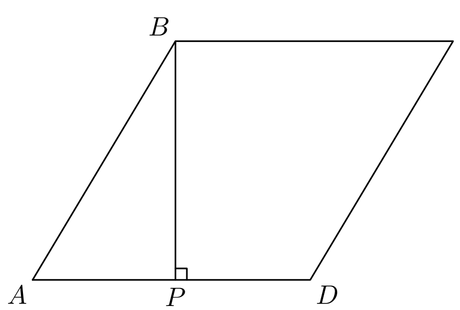
2 In rhombus $ABCD$, point P lies on segment $\overline{AD}$ such that $BP \perp AD$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$?



(A) $3\sqrt{5}$ **(B)** 10 **(C)** $6\sqrt{5}$ **(D)** 20 **(E)** 25

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3 How many of the first ten numbers of the sequence 121, 11211, 1112111, ... are prime numbers?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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4 For how many values of the constant k will the polynomial $x^2+kx+36$ have two distinct integer roots?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 14 **(E)** 16

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5 The point $(-1,-2)$ is rotated 270° counterclockwise about the point $(3,1)$. What are the coordinates of its new position?

(A) $(-3,-4)$ **(B)** $(0,5)$ **(C)** $(2,-1)$ **(D)** $(4,3)$ **(E)** $(6,-3)$

---

6 Consider the following 100 sets of 10 elements each:

$$\begin{aligned}&\{1,2,3,\cdots,10\},\\&\{11,12,13,\cdots,20\},\\&\{21,22,23,\cdots,30\},\\&\vdots\\&\{991,992,993,\cdots,1000\}.\end{aligned}$$

How many of these sets contain exactly two multiples of 7?

(A)40 **(B)**42 **(C)**42 **(D)**49 **(E)**50

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7 Camila writes down five positive integers. The unique mode of these integers is 2 greater than their median, and the median is 2 greater than their arithmetic mean. What is the least possible value for the mode?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

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8 What is the graph of $y^4+1=x^4+2y^2$ in the coordinate plane?

(A) Two intersecting parabolas **(B) Two nonintersecting parabolas** **(C) Two intersecting circles**

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9 The sequence $a_0,a_1,a_2,\cdots$ is a strictly increasing arithmetic sequence of positive integers such that

$$2^{a_7}=2^{27}\cdot a_7.$$

What is the minimum possible value of a_2?

(A)8 **(B)**12 **(C)**16 **(D)**17 **(E)**22

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10 Regular hexagon $ABCDEF$ has side length 2. Let G be the midpoint of $\overline{AB}$, and let H be the midpoint of $\overline{DE}$. What is the perimeter of $GCHF$? **(A)** $4\sqrt{3}$ **(B)** 8 **(C)** $4\sqrt{5}$ **(D)** $4\sqrt{7}$ **(E)** 12

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11 Let $f(n) = \left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$, where $i = \sqrt{-1}$. What is $f(2022)$

(A) -2 **(B)** -1 **(C)** 0 **(D)** $\sqrt{3}$ **(E)** 2

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12 Kayla rolls four fair 6-sided dice. What is the probability that at least one of the numbers Kayla rolls is greater than 4 and at least two of the numbers she rolls are greater than 2?

(A) $\frac{2}{3}$ **(B)** $\frac{19}{27}$ **(C)** $\frac{59}{81}$ **(D)** $\frac{61}{81}$ **(E)** $\frac{7}{9}$

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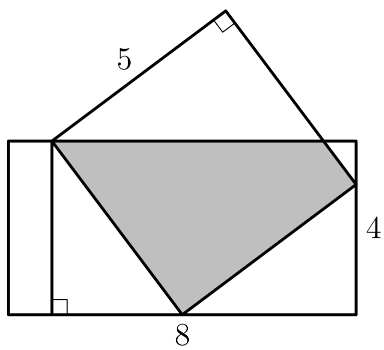
13 The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



(A) $15\frac{1}{8}$ **(B)** $15\frac{3}{8}$ **(C)** $15\frac{1}{2}$ **(D)** $15\frac{5}{8}$ **(E)** $15\frac{7}{8}$

---

14 The graph of $y = x^2 + 2x - 15$ intersects the x-axis at points A and C and the y-axis at point B. What is $\tan(\angle ABC)$?

(A) $\frac{1}{7}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{7}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{7}$

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15 One of the following numbers is not divisible by any prime number less than 10. Which is it?
(A) $2^{606} - 1$ (B) $2^{606} + 1$ (C) $2^{607} - 1$ (D) $2^{607} + 1$ (E) $2^{607} + 3^{607}$

---

16 Suppose x and y are positive real numbers such that

$x^y = 2^{64}$ and $(\log_2 x)^{\log_2 y} = 2^7$.

What is the greatest possible value of $\log_2 y$?

(A) 3 **(B)** 4 **(C)** $3+\sqrt{2}$ **(D)** $4+\sqrt{3}$ **(E)** 7

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17 How many 4×4 arrays whose entries are 0s and 1s are there such that the row sums (the sum of the entries in each row) are $1, 2, 3,$ and $4,$ in some order, and the column sums (the sum of the entries in each column) are also $1, 2, 3,$ and $4,$ in some order? For example, the array

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

satisfies the condition.

(A) 144 **(B)** 240 **(C)** 336 **(D)** 576 **(E)** 624

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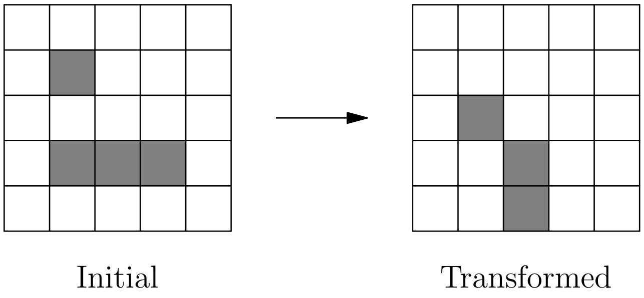
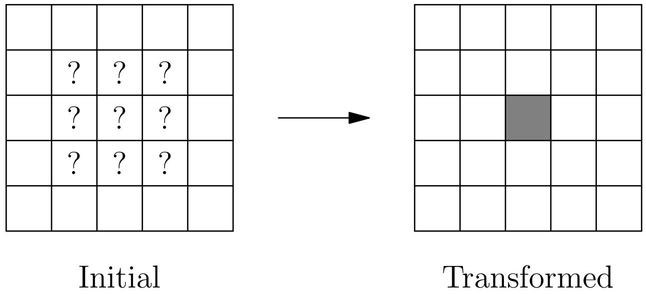
18 Each square in a 5×5 grid is either filled or empty, and has up to eight adjacent neighboring squares, where neighboring squares share either a side or a corner. The grid is transformed by the following rules:

- Any filled square with two or three filled neighbors remains filled.
- Any empty square with exactly three filled neighbors becomes a filled square.
- All other squares remain empty or become empty.

A sample transformation is shown in the figure below.



Suppose the 5×5 grid has a border of empty squares surrounding a 3×3 subgrid. How many initial configurations will lead to a transformed grid consisting of a single filled square in the center after a single transformation? (Rotations and reflections of the same configuration are considered different.)



(A) 14 (B) 18 (C) 22 (D) 26 (E) 30

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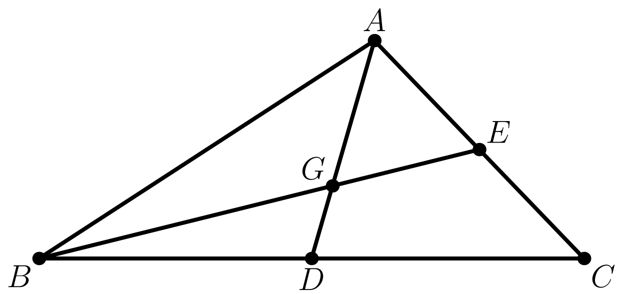
19 Don't have original wording:

In $\triangle ABC$ medians $\overline{AD}$ and $\overline{BE}$ intersect at G and $\triangle AGE$ is equilateral. Then $\cos(C)$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers and p is a positive integer not divisible by the square of any prime. What is $m+n+p$?



(A)44 **(B)**48 **(C)**52 **(D)**56 **(E)**60

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20 Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial x^2+x+1, the remainder is $x+2$, and when $P(x)$ is divided by the polynomial x^2+1, the remainder is $2x+1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

(A) 10 **(B)** 13 **(C)** 19 **(D)** 20 **(E)** 23

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21 Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations $x^2+y^2=4$, $x^2+y^2=64$, and $(x-5)^2+y^2=3$. What is the sum of the areas of all circles in S?

(A) 48π **(B)** 68π **(C)** 96π **(D)** 102π **(E)** 136π

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22 Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n = 1, 2, 3,$ Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0, 1)$. During the nth step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the nth step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

(A) $\frac{1}{3}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{5}{6}$

---

23 Let x_0, x_1, x_2, $\cdots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

(A) 6 **(B)** 7 **(C)** 12 **(D)** 14 **(E)** 15

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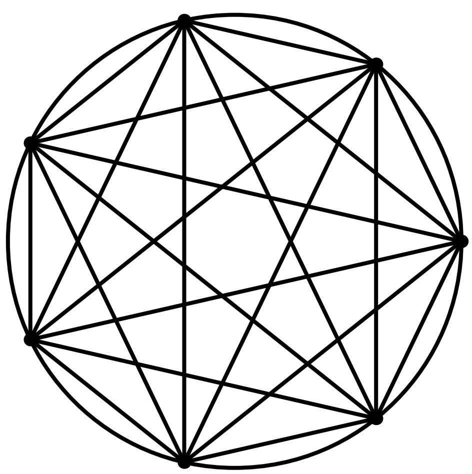
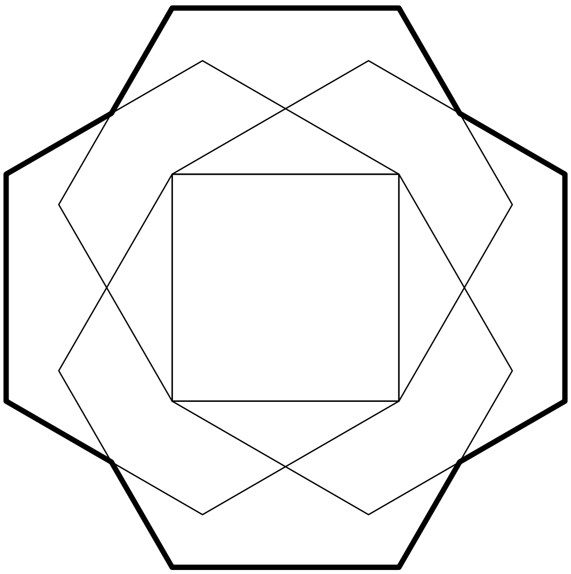
24 The figure below depicts a regular 7-gon inscribed in a unit circle.



What is the sum of the 4th powers of the lengths of all 21 of its edges and diagonals?

(A)49 **(B)**98 **(C)**147 **(D)**168 **(E)**196

25 Four regular hexagons surround a square with a side length 1, each one sharing an edge with the square, as shown in the figure below. The area of the resulting 12-sided outer nonconvex polygon can be written as $m\sqrt{n}+p$, where m, n, and p are integers and n is not divisible by the square of any prime. What is $m+n+p$?



(A) -12 **(B)** -4 **(C)**4 **(D)**24 **(E)**32