National Math Olympiad (3rd Round) 2016

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1 The sequence (a_n) is defined as:

$$a_1 = 1007$$

$$a_{i+1} \geq a_i + 1$$

Prove the inequality:

$$\frac{1}{2016} > \sum_{i=1}^{2016} \frac{1}{a_{i+1}^2 + a_{i+2}^2}$$

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2 Find all function $f : \mathbb{N} \to \mathbb{N}$ such that for all $a, b \in \mathbb{N}$, $(f(a) + b)f(a + f(b)) = (a + f(b))^2$

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3 Do there exists many infinitely points like $(x_1, y_1), (x_2, y_2), ...$ such that for any sequences like $b_1, b_2, ...$ of real numbers there exists a polynomial $P(x, y) \in R[x, y]$ such that we have for all i : $P(x_i, y_i) = b_i$

– Geometry

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1 Let ABC be an arbitrary triangle,P is the intersection point of the altitude from C and the tangent line from A to the circumcircle. The bisector of angle A intersects BC at D . PD intersects AB at K, if H is the orthocenter then prove : $HK \perp AD$

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2 Let ABC be an arbitrary triangle. Let E, E be two points on AB, AC respectively such that their distance to the midpoint of BC is equal. Let P be the second intersection of the triangles ABC, AEF circumcircles . The tangents from E, F to the circumcircle of AEF intersect each other at K. Prove that : $\angle KPA = 90$

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3 Let ABC be a triangle and let AD, BE, CF be its altitudes . FA_1, DB_1, EC_1 are perpendicular segments to BC, AC, AB respectively.
Prove that : ABC $A_1B_1C_1$

– Number Theory

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1 Let F be a subset of the set of positive integers with at least two elements and $P(x)$ be a polynomial with integer coefficients such that for any two distinct elements of F like a and b, the following two conditions hold

- $a+b \in F$, and
- $\gcd(P(a), P(b)) = 1$.

Prove that $P(x)$ is a constant polynomial.

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2 Let P be a polynomial with integer coefficients. We say P is *good* if there exist infinitely many prime numbers q such that the set

$$X = \{P(n) \mod q : \quad n \in \mathbb{N}\}$$

has at least $\frac{q+1}{2}$ members.
Prove that the polynomial $x^3 + x$ is good.

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3 Let m be a positive integer. The positive integer a is called a *golden residue* modulo m if $\gcd(a, m) = 1$ and $x^x \equiv a \pmod{m}$ has a solution for x. Given a positive integer n, suppose that a is a golden residue modulo n^n. Show that a is also a golden residue modulo n^{n^n}.

Proposed by Mahyar Sefidgaran

– Combinatorics

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1 Find the number of all permutations of $\{1, 2, \cdots, n\}$ like p such that there exists a unique $i \in \{1, 2, \cdots, n\}$ that :

$$p(p(i)) \geq i$$

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2 Is it possible to divide a 7×7 table into a few connected parts of cells with the same perimeter?
(A group of cells is called connected if any cell in the group, can reach other cells by passing through the sides of cells.)

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3 There are 24 robots on the plane. Each robot has a 70° field of view. What is the maximum number of observing relations?
(Observing is a one-sided relation)

– Algebra

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1 Let $P(x) \in \mathbb{Z}[X]$ be a polynomial of degree 2016 with no rational roots. Prove that there exists a polynomial $T(x) \in \mathbb{Z}[X]$ of degree 1395 such that for all distinct (not necessarily real) roots of $P(x)$ like (α, β) :

$$T(\alpha) - T(\beta) \notin \mathbb{Q}$$

Note: $\mathbb{Q}$ is the set of rational numbers.

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2 Let $a,b,c \in \mathbb{R}^+$ and $abc=1$ prove that:

$\frac{a+b}{(a+b+1)^2}+\frac{b+c}{(b+c+1)^2}+\frac{c+a}{(c+a+1)^2} \geq \frac{2}{a+b+c}$

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3 Find all functions $f:\mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers x,y :

$$f(y)f(x+f(y))=f(x)f(xy)$$

– Geometry

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1 In triangle ABC , w is a circle which passes through B,C and intersects AB,AC at E,F respectively. BF,CE intersect the circumcircle of ABC at B',C' respectively. Let A' be a point on BC such that $\angle C'A'B=\angle B'A'C$.
Prove that if we change w, then all the circumcircles of triangles $A'B'C'$ passes through a common point.

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2 Given $\triangle ABC$ inscribed in (O) an let I and I_a be it's incenter and A-excenter ,respectively.
Tangent lines to (O) at C,B intersect the angle bisector of A at M,N ,respectively.
Second tangent lines through M,N intersect (O) at X,Y.

Prove that $XYII_a$ is cyclic.

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3 Given triangle $\triangle ABC$ and let D,E,F be the foot of angle bisectors of A,B,C ,respectively. M,N lie on EF such that $AM=AN$. Let H be the foot of A-altitude on BC.
Points K,L lie on EF such that triangles $\triangle AKL,\triangle HMN$ are correspondingly similiar (with the given order of vertices) such that $AK \not\parallel HM$ and $AK \not\parallel HN$.

Show that: $DK=DL$

– Number Theory

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1 Let p,q be prime numbers (q is odd). Prove that there exists an integer x such that:

$$q|(x+1)^p-x^p$$

If and only if

$$q \equiv 1 \pmod p$$

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2 We call a function g *special* if $g(x) = a^{f(x)}$ (for all x) where a is a positive integer and f is polynomial with integer coefficients such that $f(n) > 0$ for all positive integers n.

A function is called an *exponential polynomial* if it is obtained from the product or sum of special functions. For instance, $2^x 3^{x^2+x-1} + 5^{2x}$ is an exponential polynomial.

Prove that there does not exist a non-zero exponential polynomial $f(x)$ and a non-constant polynomial $P(x)$ with integer coefficients such that

$$P(n)|f(n)$$

for all positive integers n.

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3 A sequence $P = \{a_n\}$ is called a Permutation of natural numbers (positive integers) if for any natural number m, there exists a unique natural number n such that $a_n = m$.

We also define $S_k(P)$ as: $S_k(P) = a_1 + a_2 + \cdots + a_k$ (the sum of the first k elements of the sequence).

Prove that there exists infinitely many distinct Permutations of natural numbers like $P_1, P_2, \cdots$ such that:

$$\forall k, \forall i < j : S_k(P_i)|S_k(P_j)$$

– Combinatorics

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1 In an election, there are 1395 candidates and some voters. Each voter, arranges all the candidates by the priority order.

We form a directed graph with 1395 vertices, an arrow is directed from U to V when the candidate U is at a higher level of priority than V in more than half of the votes. (otherwise, there's no edge between U, V)

Is it possible to generate all complete directed graphs with 1395 vertices?

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2 A 100×100 table is given. At the beginning, every unit square has number $"0"$ written in them. Two players playing a game and the game stops after 200 steps (each player plays 100 steps).

In every step, one can choose a row or a column and add 1 to the written number in all of it's squares $\pmod 3$.
First player is the winner if more than half of the squares (5000 squares) have the number $"1"$ written in them,
Second player is the winner if more than half of the squares (5000 squares) have the number $"0"$ written in them. Otherwise, the game is draw.

Assume that both players play at their best. What will be the result of the game ?

Proposed by Mahyar Sefidgaran

3 A 30×30 table is given. We want to color some of it's unit squares such that any colored square has at most k neighbors. (Two squares (i,j) and (x,y) are called neighbors if $i-x, j-y \equiv 0,-1,1 \pmod{30}$ and $(i,j) \neq (x,y)$. Therefore, each square has exactly 8 neighbors)
What is the maximum possible number of colored squares if:

$a) k=6$

$b) k=1$