ICMC 2019-2020

www.artofproblemsolving.com/community/c3210286

by mastermind.hk16, sjbs

Round 1 24 November 2019

1 Alice and Bob play a game on a sphere which is initially marked with a finite number of points. Alice and Bob then take turns making moves, with Alice going first:

- On Alice's move, she counts the number of marked points on the sphere, n. She then marks another $n+1$ points on the sphere.

- On Bob's move, he chooses one hemisphere and removes all marked points on that hemisphere, including any marked points on the boundary of the hemisphere.

Can Bob always guarantee that after a finite number of moves, the sphere contains no marked points?

(A *hemisphere* is the region on a sphere that lies completely on one side of any plane passing through the centre of the sphere.)

proposed by the ICMC Problem Committee

2 Find integers a and b such that

$$a^b = 3^0\binom{2020}{0} - 3^1\binom{2020}{2} + 3^2\binom{2020}{4} - \cdots + 3^{1010}\binom{2020}{2020}.$$

proposed by the ICMC Problem Committee

3 Consider a grid of points where each point is coloured either white or black, such that no two rows have the same sequence of colours and no two columns have the same sequence of colours. Let a *table* denote four points on the grid that form the vertices of a rectangle with sides parallel to those of the grid. A table is called *balanced* if one diagonal pair of points are coloured white and the other diagonal pair black.

Determine all possible values of $k \geq 2$ for which there exists a colouring of a $k \times 2019$ grid with no balanced tables.

proposed by the ICMC Problem Committee

4 Let n be a non-negative integer. Define the *decimal digit product* $D(n)$ inductively as follows:

- If n has a single decimal digit, then let $D(n) = n$.

- Otherwise let $D(n) = D(m)$, where m is the product of the decimal digits of n.

Let $P_k(1)$ be the probability that $D(i) = 1$ where i is chosen uniformly randomly from the set of integers between 1 and k (inclusive) whose decimal digit products are not 0.

Compute $\lim_{k\to\infty} P_k(1)$.

proposed by the ICMC Problem Committee

---

5 A particle moves from the point P to the point Q in the Cartesian plane. When it passes through any point (x, y), the particle has an instantaneous speed of $\sqrt{x+y}$. Compute the minimum time required for the particle to move:

(i) from $P_1 = (-1, 0)$ to $Q_1 = (1, 0)$, and

(ii) from $P_2 = (0, 1)$ to $Q_2 = (1, 1)$.

proposed by the ICMC Problem Committee

---

6 Let $\varepsilon < \frac{1}{2}$ be a positive real number and let U_ε denote the set of real numbers that differ from their nearest integer by at most ε. Prove that there exists a positive integer m such that for any real number x, the sets $\{x, 2x, 3x, ..., mx\}$ and U_ε have at least one element in common.

proposed by the ICMC Problem Committee

Round 2 23 February 2020

---

1 An *automorphism* of a group $(G, *)$ is a bijective function $f : G \to G$ satisfying $f(x * y) = f(x) * f(y)$ for all $x, y \in G$.
Find a group $(G, *)$ with fewer than $(201.6)^2 = 40642.56$ unique elements and exactly 2016^2 unique automorphisms.

Proposed by the ICMC Problem Committee

---

2 Let $\mathbb{R}^2$ denote the set of points in the Euclidean plane. For points $A, P \in \mathbb{R}^2$ and a real number k, define the *dilation* of A about P by a factor of k as the point $P+k(A-P)$. Call a sequence of point $A_0, A_1, A_2, \ldots \in \mathbb{R}^2$ *unbounded* if the sequence of lengths $|A_0 - A_0|, |A_1 - A_0|, |A_2 - A_0|, \ldots$ has no upper bound.
Now consider n distinct points $P_0, P_1, \ldots, P_{n-1} \in \mathbb{R}^2$, and fix a real number r. Given a starting point $A_0 \in \mathbb{R}^2$, iteratively define A_{i+1} by dilating A_i about P_j by a factor of r, where j is the remainder of i when divided by n.

Prove that if $|r| \geq 1$, then for any starting point $A_0 \in \mathbb{R}^2$, the sequence $A_0, A_1, A_2, \ldots$ is either periodic or unbounded.

Proposed by the ICMC Problem Committee

---

3 Let $\mathbb{R}$ denote the set of real numbers. A subset $S \subseteq \mathbb{R}$ is called *dense* if any non-empty open interval of $\mathbb{R}$ contains at least one element in S. For a function $f : \mathbb{R} \to \mathbb{R}$, let $\mathcal{O}_f(x)$ denote the set $\{x, f(x), f(f(x)), \ldots\}$.

(a) Is there a function $g : \mathbb{R} \to \mathbb{R}$, continuous everywhere in $\mathbb{R}$ such that $\mathcal{O}_g(x)$ is dense for all $x \in \mathbb{R}$ for all $x \in \mathbb{R}$?

(b) Is there a function $h : \mathbb{R} \to \mathbb{R}$, continuous at all but a single $x_0 \in \mathbb{R}$, such that $\mathcal{O}_h(x)$ is dense for all $x \in \mathbb{R}$?

Proposed by the ICMC Problem Committee

---

4 Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a set of $n \geq 2020$ distinct points on the Euclidean plane, no three of which are collinear. Andy the ant starts at some point S_{i_1} in $\mathcal{S}$ and wishes to visit a series of 2020 points $\{S_{i_1}, S_{i_2}, \ldots, S_{i_{2020}}\} \subseteq \mathcal{S}$ in order, such that $i_j > i_k$ whenever $j > k$. It is known that ants can only travel between points in $\mathcal{S}$ in straight lines, and that an ant's path can never self-intersect.

Find a positive integer n such that Andy can always fulfill his wish.

(Lower n will be awarded more marks. Bounds for this problem may be used as a tie-breaker, should the need to do so arise.)

Proposed by the ICMC Problem Committee
