ICMC 2022-2023

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by mastermind.hk16

Round 1 27 November 2022

1 Two straight lines divide a square of side length 1 into four regions. Show that at least one of the regions has a perimeter greater than or equal to 2.

Proposed by Dylan Toh

2 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) > f(x) > 0$ for all real numbers x. Show that $f(8) > 2022f(0)$.

Proposed by Ethan Tan

3 Bugs Bunny plays a game in the Euclidean plane. At the n-th minute $(n \geq 1)$, Bugs Bunny hops a distance of F_n in the North, South, East, or West direction, where F_n is the n-th Fibonacci number (defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$). If the first two hops were perpendicular, prove that Bugs Bunny can never return to where he started.

Proposed by Dylan Toh

4 Let $\mathcal{G}$ be a simple graph with n vertices and m edges such that no two cycles share an edge. Prove that $2m < 3n$.

Note: A *simple graph* is a graph with at most one edge between any two vertices and no edges from any vertex to itself. A *cycle* is a sequence of distinct vertices $v_1, \dots, v_n$ such that there is an edge between any two consecutive vertices, and between v_n and v_1.

Proposed by Ethan Tan

5 Let $[0, 1]$ be the set $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Does there exist a continuous function $g : [0, 1] \to [0, 1]$ such that no line intersects the graph of g infinitely many times, but for any positive integer n there is a line intersecting g more than n times?

Proposed by Ethan Tan

6 Consider the sequence defined by $a_1 = 2022$ and $a_{n+1} = a_n + e^{-a_n}$ for $n \geq 1$. Prove that there exists a positive real number r for which the sequence

$$\{ra_1\}, \{ra_{10}\}, \{ra_{100}\}, \ldots$$

converges.

Note: $\{x\} = x - \lfloor x \rfloor$ denotes the part of x after the decimal point.

Proposed by Ethan Tan

Round 2 26 February 2023

1 The city of Atlantis is built on an island represented by $[-1, 1]$, with skyline initially given by $f(x) = 1 - |x|$. The sea level is currently $y = 0$, but due to global warming, it is rising at a rate of 0.01 a year. For any position $-1 < x < 1$, while the building at x is not completely submerged, then it is instantaneously being built upward at a rate of r per year, where r is the distance (along the x-axis) from this building to the nearest completely submerged building.
How long will it be until Atlantis becomes completely submerged?

Proposed by Ethan Tan

2 Show that if the distance between opposite edges of a tetrahedron is at least 1, then its volume is at least $\frac{1}{3}$.

Proposed by Simeon Kiflie

3 The numbers $1, 2, \ldots, n$ are written on a blackboard and then erased via the following process:
- Before any numbers are erased, a pair of numbers is chosen uniformly at random and circled.
- Each minute for the next $n - 1$ minutes, a pair of numbers still on the blackboard is chosen uniformly at random and the smaller one is erased.
- In minute n, the last number is erased.

What is the probability that the smaller circled number is erased before the larger?

Proposed by Ethan Tan

4 Do there exist infinitely many positive integers m such that the sum of the positive divisors of m (including m itself) is a perfect square?

Proposed by Dylan Toh

5 A clock has an hour, minute, and second hand, all of length 1. Let T be the triangle formed by the ends of these hands. A time of day is chosen uniformly at random. What is the expected value of the area of T?

Proposed by Dylan Toh