Brazilian Undergraduate Math Olympiad 2022

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– Day 1

1 Let $0 < a < 1$. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous at $x = 0$ such that $f(x) + f(ax) = x, \forall x \in \mathbb{R}$

2 Let G be the set of 2×2 matrices that such

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \text{ is a multiple of } 3 \right\}$$

and two matrices in G:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}$$

Show that any matrix in G can be written as a product $M_1 M_2 \cdots M_r$ such that $M_i \in \{A, A^{-1}, B, B^{-1}\}, \forall i \leq r$

3 Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of integers. Define $a_n^{(0)} = a_n$ for all $n \in \mathbb{N}$. For all $M \geq 0$, we define $(a_n^{(M+1)})_{n \in \mathbb{N}} : a_n^{(M+1)} = a_{n+1}^{(M)} - a_n^{(M)}, \forall n \in \mathbb{N}$. We say that $(a_n)_{n \in \mathbb{N}}$ is $(\text{M}+1)\text{-self-referencing}$ if there exists k_1 and k_2 fixed positive integers such that $a_{n+k_1} = a_{n+k_2}^{(M+1)}, \forall n \in \mathbb{N}$.

(a) Does there exist a sequence of integers such that the smallest M such that it is $\text{M-self-referencing}$ is $M = 2022$?

(a) Does there exist a stricly positive sequence of integers such that the smallest M such that it is $\text{M-self-referencing}$ is $M = 2022$?

– Day 2

4 Let $\alpha, c > 0$, define $x_1 = c$ and let $x_{n+1} = x_n e^{-x_n^\alpha}$ for $n \geq 1$. For which values of β does $\sum_{i=1}^{\infty} x_n^\beta$ converge?

5 Given $X \subset \mathbb{N}$, define $d(X)$ as the largest $c \in [0, 1]$ such that for any $a < c$ and $n_0 \in \mathbb{N}$, there exists $m, r \in \mathbb{N}$ with $r \geq n_0$ and $\frac{|X \cap [m, m+r)|}{r} \geq a$.

Let $E, F \subset \mathbb{N}$ such that $d(E)d(F) > 1/4$. Prove that for any prime p and $k \in \mathbb{N}$, there exists $m \in E, n \in F$ such that $m \equiv n \pmod{p^k}$

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6 Let $p \equiv 3 \pmod 4$ be a prime and θ some angle such that $\tan(\theta)$ is rational. Prove that $\tan((p+1)\theta)$ is a rational number with numerator divisible by p, that is, $\tan((p+1)\theta) = \frac{u}{v}$ with $u, v \in \mathbb{Z}, v > 0, \operatorname{mdc}(u, v) = 1$ and $u \equiv 0 \pmod p$.

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