Dutch IMO Team Selection Test 2022

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by parmenides51, FedeX333X, k.vasilev, Kamran011, pinetree1

– Day 1

1 Determine all positive integers $n \geq 2$ which have a positive divisor $m|n$ satisfying

$$n = d^3 + m^3.$$

where d is the smallest divisor of n which is greater than 1.

2 Two circles Γ_1 and $\Gamma_2$are given with centres O_1 and O_2 and common exterior tangents ℓ_1 and ℓ_2. The line ℓ_1 intersects Γ_1 in A and Γ_2 in B. Let X be a point on segment O_1O_2, not lying on Γ_1 or Γ_2. The segment AX intersects Γ_1 in $Y \neq A$ and the segment BX intersects Γ_2 in $Z \neq B$. Prove that the line through Y tangent to Γ_1 and the line through Z tangent to Γ_2 intersect each other on ℓ_2.

3 For real numbers x and y we define $M(x, y)$ to be the maximum of the three numbers xy, $(x-1)(y-1)$, and $x+y-2xy$. Determine the smallest possible value of $M(x, y)$ where x and y range over all real numbers satisfying $0 \leq x, y \leq 1$.

4 In a sequence $a_1, a_2, ..., a_{1000}$ consisting of 1000 distinct numbers a pair (a_i, a_j) with $i < j$ is called *ascending* if $a_i < a_j$ and *descending* if $a_i > a_j$. Determine the largest positive integer k with the property that every sequence of 1000 distinct numbers has at least k non-overlapping ascending pairs or at least k non-overlapping descending pairs.

– Day 2

1 Consider an acute triangle ABC with $|AB| > |CA| > |BC|$. The vertices D, E, and F are the base points of the altitudes from A, B, and C, respectively. The line through F parallel to DE intersects BC in M. The angular bisector of $\angle MFE$ intersects DE in N. Prove that F is the circumcentre of $\triangle DMN$ if and only if B is the circumcentre of $\triangle FMN$.

2 Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

Let $\lambda \geq 1$ be a real number and n be a positive integer with the property that $\lfloor \lambda^{n+1} \rfloor, \lfloor \lambda^{n+2} \rfloor, \cdots, \lfloor \lambda^{4n} \rfloor$ are all perfect squares. Prove that $\lfloor \lambda \rfloor$ is a perfect square.

3 There are 15 lights on the ceiling of a room, numbered from 1 to 15. All lights are turned off. In another room, there are 15 switches: a switch for lights 1 and 2, a switch for lights 2 and 3, a switch for lights 3 en 4, etcetera, including a sqitch for lights 15 and 1. When the switch for such a pair of lights is turned, both of the lights change their state (from on to off, or vice versa). The switches are put in a random order and all look identical. Raymond wants to find out which switch belongs which pair of lights. From the room with the switches, he cannot see the lights. He can, however, flip a number of switches, and then go to the other room to see which lights are turned on. He can do this multiple times. What is the minimum number of visits to the other room that he has to take to determine for each switch with certainty which pair of lights it corresponds to?

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4 Determine all positive integers $d,$ such that there exists an integer $k \geq 3,$ such that One can arrange the numbers $d, 2d, \ldots, kd$ in a row, such that the sum of every two consecutive of them is a perfect square.

– Day 3

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1 Find all quadruples (a, b, c, d) of non-negative integers such that $ab = 2(1 + cd)$ and there exists a non-degenerate triangle with sides of length $a - c$, $b - d$, and $c + d$.

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2 Let $n > 1$ be an integer. There are n boxes in a row, and there are $n + 1$ identical stones. A *distribution* is a way to distribute the stones over the boxes, in which every stone is in exactly one of the boxes. We say that two of such distributions are a *stone's throw away* from each other if we can obtain one distribution from the other by moving exactly one stone from one box to another. The *cosiness* of a distribution a is defined as the number of distributions that are a stone's throw away from a. Determine the average cosiness of all possible distributions.

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3 Let n be a natural number. An integer $a > 2$ is called n-decomposable, if $a^n - 2^n$ is divisible by all the numbers of the form $a^d + 2^d$, where $d \neq n$ is a natural divisor of n. Find all composite $n \in \mathbb{N}$, for which there's an n-decomposable number.

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4 Let ABC be a triangle with a right angle at C. Let I be the incentre of triangle ABC, and let D be the foot of the altitude from C to AB. The incircle ω of triangle ABC is tangent to sides BC, CA, and AB at A_1, B_1, and C_1, respectively. Let E and F be the reflections of C in lines C_1A_1 and C_1B_1, respectively. Let K and L be the reflections of D in lines C_1A_1 and C_1B_1, respectively.

Prove that the circumcircles of triangles A_1EI, B_1FI, and C_1KL have a common point.
