Math Prize for Girls Olympiad 2022

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by Ravi B

1 Let a, b, c be positive integers with $a \leq 10$. Suppose the parabola $y = ax^2 + bx + c$ meets the x-axis at two distinct points A and B. Given that the length of $\overline{AB}$ is irrational, determine, with proof, the smallest possible value of this length, across all such choices of (a, b, c).

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2 Determine, with proof, whether or not there exists a *non-isosceles* trapezoid $ABCD$ such that the lengths AC and BD both lie in the set $\{DA + AB, AB + BC, BC + CD, CD + DA, AB + CD, BC + DA\}$.

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3 Serena has written 20 copies of the number 1 on a board. In a move, she is allowed to
$*$ erase two of the numbers and replace them with their sum, or
$*$ erase one number and replace it with its reciprocal.
Whenever a fraction appears on the board, Serena writes it in simplest form. Prove that Serena can never write a fraction less than 1 whose numerator is over 9000, regardless of the number of moves she makes.

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4 Let $n > 1$ be an integer. Let A denote the set of divisors of n that are less than $\sqrt{n}$. Let B denote the set of divisors of n that are greater than $\sqrt{n}$. Prove that there exists a bijective function $f\colon A \to B$ such that a divides $f(a)$ for all $a \in A$.

(We say f is *bijective* if for every $b \in B$ there exists a unique $a \in A$ with $f(a) = b$.)
