Spain Mathematical Olympiad 1968

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by parmenides51

– Day 1

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1 In one night the air temperature remained constant, several degrees below zero, and that of the water of a very extensive cylindrical pond, which formed a layer 10 cm deep, it reached zero degrees, beginning then to form a layer of ice on the surface. Under these conditions it can be assumed that the thickness of the ice sheet formed is directly proportional to the square root of the time elapsed. At 0 h, the thickness of the ice was 3 cm and at 4 h it was just over to freeze the water in the pond. Calculate at what time the ice sheet began to form, knowing that the density of the ice formed was 0.9.

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2 Justify if continuity can be affirmed, denied or cannot be decided in the point$x = 0$ of a real function $f(x)$ of real variable, in each of the three (independent) cases .

a) It is known only that for all natural n: $f\left(\frac{1}{2n}\right) = 1$ and $f\left(\frac{1}{2n+1}\right) = -1$.

b) It is known that for all nonnegative real x is $f(x) = x^2$ and for negative real x is $f(x) = 0$.

c) It is only known that for all natural n it is $f\left(\frac{1}{n}\right) = 1$.

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3 Given a square whose side measures a, consider the set of all points of its plane through which passes a circumference of radius whose circle contains to the quoted square. You are asked to prove that the contour of the figure formed by the points with this property is formed by arcs of circumference, and determine the positions, their centers, their radii and their lengths.

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4 At the two ends A, B of a diameter (of length $2r$) of a pavement horizontal circular rise two vertical columns, of equal height h, whose ends support a beam $A'B'$ of length equal to the before mentioned diameter. It forms a covered by placing numerous taut cables (which are admitted to be rectilinear), joining points of the beam $A'B'$ with points of the circumference edge of the pavement, so that the cables are perpendicular to the beam $A'B'$. You want to find out the volume enclosed between the roof and the pavement.

En los dos extremos A, B de un di´ametro (de longitud 2r) de un pavimento circular horizontal se levantan sendas columnas verticales, de igual altura h, cuyos extremos soportan una viga A’ B’ de longitud igual al diametro citado. Se forma una cubierta colocando numerosos cables tensos (que se admite que quedan rectilıneos), uniendo puntos de la viga A’B’ con puntos de la circunferencia borde del pavimento, de manera que los cables queden perpendiculares a la viga A’B’ . Se desea averiguar el volumen encerrado entre la cubierta y el pavimento.

– Day 2

5 Find the locus of the center of a rectangle, whose four vertices lies on the sides of a given triangle.

6 Check and justify , if in every tetrahedron are concurrent:
a) The perpendiculars to the faces at their circumcenters.
b) The perpendiculars to the faces at their orthocenters.
c) The perpendiculars to the faces at their incenters.
If so, characterize with some simple geometric property the point in that attend If not, show an example that clearly shows the not concurrency.

7 In the sequence of powers of 2 (written in the decimal system, beginning with $2^1 = 2$) there are three terms of one digit, another three of two digits, another three of 3, four out of 4, three out of 5, etc. Clearly reason the answers to the following questions:
a) Can there be only two terms with a certain number of digits?
b) Can there be five consecutive terms with the same number of digits?
c) Can there be four terms of n digits, followed by four with $n + 1$ digits?
d) What is the maximum number of consecutive powers of 2 that can be found without there being four among them with the same number of digits?

8 We will assume that the sides of a square are reflective and we will designate them with the names of the four cardinal points. Marking a point on the side N , determine in which direction a ray of light should exit (into the interior of the square) so that it returns to it after having undergone n reflections on the side E , another n on the side W , m on the S and $m - 1$ on the N, where n and m are known natural numbers. What happens if m and n are not prime to each other? Calculate the length of the light ray considered as a function of m and n, and of the length of the side of the square.